Lecture 1: Affine varieties – Exercamples

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1 Varieties and ideals

Exercample 1.

- Show that $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle_R)$, where $\langle \mathcal{F} \rangle_R \coloneqq \{ \sum_i g_i f_i \mid g_i \in R,\ f_i \in \mathcal{F} \}$
- If $X, Y \subseteq \mathbb{A}^n$ are varieties, then so are $X \cap Y$ and $X \cup Y$.

Exercample 2. Show that for all $\mathcal{F}$, X

- $\mathbb{V}(\dots)$ and $I(\dots)$ are *antitone*: $\mathcal{F} \subseteq \mathcal{G} \implies \mathbb{V}(\mathcal{F}) \supseteq \mathbb{V}(\mathcal{G})$
- $\mathcal{F} \subseteq I(\mathbb{V}(\mathcal{F}))$ and $I(\mathbb{V}(I(X))) = I(X)$
- $X \subseteq \mathbb{V}(I(X))$ and $\mathbb{V}(I(\mathbb{V}(\mathcal{F}))) = \mathbb{V}(\mathcal{F})$
- What is $I(p)$, $p = (p_1, \dots, p_n) \in \mathbb{A}^n$? (Hint: Taylor expansion)

Exercample 3.

- Every open set of X is a union of $D_X(f) = \{ x \in X \mid f(x) \neq 0 \}$, $f \in R$
- Y is dense in X if and only if every non-empty open set of X meets Y
- X is irreducible iff every non-empty open set is dense.
- If X is a variety and $Y \subseteq X$, then Y is irreducible iff $\overline{Y}$ is irreducible
- $X = \bigcup_{i=1}^{c} X_i$ uniquely for X_i closed irreducible, $X_i \not\subseteq X_j$ for $i \neq j$ ("components")

Exercample 4. Let $I \subseteq R$ be an ideal.

- $\sqrt{I}$ is an ideal
- $I(X)$ is a radical ideal.
- $I \subseteq R$ maximal $\implies$ I prime $\implies$ I radical.

2 The Nullstellensatz

Exercample 5. Let $I \subseteq R$ be an ideal such that $\mathbb{V}(I) = \{\mathbf{0}\}$ and let $g \in R$.

- Show that $\ell \coloneqq \dim_{\mathbb{C}} R/I < \infty$.
- Show that the linear map $\cdot g \colon R/I \to R/I$ has characteristic polynomial $(t - g(\mathbf{0}))^{\ell}$.

Exercample 6. Describe the vanishing ideal of the following variety of skew-symmetric matrices:

$$\left\{ A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \,\middle|\, \operatorname{rank} A < 4 \right\} \subseteq \mathbb{A}^6.$$

3 Morphisms

Exercample 7.

- Morphisms $X \to \mathbb{A}^1 = \mathbb{C}$ are in bijection with $R/I(X)$
- Show that a morphism $X \to Y$ is *continuous*: Preimages of open/closed sets from Y are open/closed in X
- Describe the image of $\phi \colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$

Exercample 8.

- If $X, Y \subseteq \mathbb{A}^n$ are such that $X \cap Y = \emptyset$, show that $\mathcal{O}(X \cup Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$. (Hint: Chinese remainder theorem).
- What is the coordinate ring of a set of points?
- Show that a morphism $\phi \colon X \to Y$ induces a $\mathbb{C}$-algebra homomorphism $\phi^* \colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Is this correspondence reversible?
- Show that $X = \mathbb{V}(xy - 1)$ is not *isomorphic* to $\mathbb{A}^1$ (no mutually-inverse morphisms).

4 Dimension and fiber dimension

Exercample 9.

- Let $X \subseteq \mathbb{A}^n$ be a variety, and $X = X_1 \cup \cdots \cup X_r$ for subvarieties X_i. Show that $\dim X = \max\{\dim X_1, \ldots, \dim X_r\}$
- If $X \to Y$ is dominant, show that $\dim X \geq \dim Y$
- Show $\dim \mathbb{A}^1 = 1$. Can you argue $\dim \mathbb{A}^2 = 2$? $\dim \mathbb{A}^3 = 3$?

Exercample 10.

- Which result in linear algebra is the fiber dimension theorem generalizing?

- Find an example where equality does not always hold.
- What is the dimension of the variety of $m \times n$-matrices of rank $\leq r$.

5 How to show it's reduced?

Exercample 11.

- Show that the smooth locus $X^{\text{sm}} \subseteq X$ is dense open for $X = \mathbb{V}(f)$.
- Find the smooth locus of $\mathbb{V}(y^2 - x^3)$

Exercample 12. Let $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1, x_2, y_1, y_2]$. Show that $\mathbb{V}(\mathcal{F})$ is a smooth irreducible variety of dimension 2. What do points $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$ "represent"?