Lecture 2: Projective varieties – Exercamples

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Exercample 1.

- Which result in linear algebra is the fiber dimension theorem generalizing?
- Find an example where equality does not always hold for all $y \in Y$.
- What is the dimension of the variety of $m \times n$-matrices of rank $\leq r$?

Exercample 2. Show that any lines $L_1, L_2 \subseteq \mathbb{P}^2$ intersect ($L_i = \mathbb{V}(\ell_i)$, $0 \neq \ell_i \in S_1$).

Exercample 3.

- $I \subseteq S$ is homogeneous iff $f \in I$ implies $f_j \in I$ for its graded comp. $f = \sum_{j=0}^{\deg f} f_j$
- $I(X)$ (homogeneous vanishing ideal) $= I(\widehat{X})$ (ideal of cone)

Exercample 4. Let $f_1, \dots, f_n \in S$ be homogeneous. Show that $X = \mathbb{V}_{\mathbb{P}^n}(f_1, \dots, f_n) \neq \emptyset$.
Show that if X is finite, then $\dim \mathbb{V}(f_1, \dots, f_k) = n - k$ for $k = 1, \dots, n$.
Show that this is generally *fails* in $\mathbb{A}^3$.

Exercample 5.

- Let $\phi \colon X \to Y$ be a morphism of affine varieties. If X is irreducible and ϕ is surjective, then Y is surjective.
- Show that the set of matrices $X \subseteq \mathbb{C}^{m \times n}$ of rank $\leq r$ is an irreducible variety
- Compute the dimension of X.

Exercample 6.

- Let $X = \left\{ (t, t^2, t^3) \,\middle|\, t \in \mathbb{C} \right\} \subseteq \mathbb{A}^3$. Show that $X = \mathbb{V}(x_1^2 - x_2, x_1^3 - x_3)$. We will see later that these equations generate $I = I(X) \subseteq R$.
- Describe $\overline{X} \subseteq \mathbb{P}^3$. Can you find generators of $I(\overline{X}) = I^{\mathrm{h}}$ up to radical?

- Compute the number of intersection points of X with a general plane $H \subseteq \mathbb{P}^3$.

Exercample 7.
- Compute the Hilbert function and degree of a hypersurface $\mathbb{V}(f) \subseteq \mathbb{P}^n$, $f \in S_d$.
- Show that the Hilbert function of a set of points $\{p_1, \ldots, p_r\} \subseteq \mathbb{P}^n$ is eventually constant r.

Exercample 8.
- Compute the Hilbert function of a hypersurface $\mathbb{V}(f)$, $f \in S_d$
- Compute the Hilbert function for a set of 3 points in $\mathbb{P}^2$

Exercample 9.
- What is the Hilbert polynomial of $\mathbb{P}^n$? Of a hypersurface?
- Show that if $X \cap Y = \emptyset$, then $P_{X \cup Y} = P_X + P_Y$, but equality may not hold for Hilbert functions.

Exercample 10. Explore the theorem for the BKK-general system $\mathcal{F} = \{-1 + x - y + xy, 2 + x + \frac{1}{2}y - xy\}$. What are its roots in $\mathbb{A}^2$? In $\mathbb{P}^2$?

Exercample 11.
- Show that this agrees with your experience from multiplicity of univariate polynomials
- Let $p = \mathbf{0} \in \mathbb{A}^3$. Compute $\text{mult}_{\mathbf{0}}(x^3, y^3, z^3)$ and $\text{mult}_{\mathbf{0}}(\text{all mon's of deg. } 3)$
- Let $f = y^2 - x^2(x+1)$ and $g = y$. Compute $\mathbb{V}(f, g)$ and the intersection multiplicities

Exercample 12. If $\mathbb{V}(I) \subseteq \mathbb{P}^n$ is a finite set $p_1, \ldots, p_r$, then $\text{hf}_{S/I}(t)$ eventually stabilizes at the value $m = \sum_{i=1}^r \text{mult}_{p_i}(I)$