Lecture 3: Multiplicities and reducedness – Exercamples

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Exercample 1.

- Show that the definition of intersection multiplicity agrees with your experience from multiplicity of univariate polynomials.
- Let $p = \mathbf{0} \in \mathbb{A}^3$. Compute $\mathrm{mult}_{\mathbf{0}}(x^3, y^3, z^3)$ and $\mathrm{mult}_{\mathbf{0}}(\text{all mon's of deg. 3})$.
- Let $f = y^2 - x^2(x+1)$ and $g = y$. Compute $\mathbb{V}(f, g)$ and the intersection multiplicities.

Exercample 2. If $\mathbb{V}(I) \subseteq \mathbb{P}^n$ is a finite set $p_1, \ldots, p_r$, then $\mathrm{hf}_{S/I}(t)$ eventually stabilizes at the value $m = \sum_{i=1}^r \mathrm{mult}_{p_i}(I)$.

Exercample 3. Consider $\mathcal{F} = \{xy, y^2\} \subseteq \mathbb{C}[x, y]$. What is $X = \mathbb{V}(\mathcal{F})$ and $I(X)$? Where does $\mathcal{F}$ define X in a reduced way?

Exercample 4.

- Show that the smooth locus is open and dense for $X = \mathbb{V}(f)$.
- Find the singular locus of $\mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ and $\mathbb{V}(xyz + xyw + xwz + yzw) \subseteq \mathbb{P}^3$.

Exercample 5. Let $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1, x_2, y_1, y_2]$. Show that $\mathbb{V}(\mathcal{F})$ is a smooth irreducible variety of dimension 2. What do points $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$ "represent"?

Exercample 6.

- Use this to argue that the degree of a variety is well-defined.
- Show that if $f \in \mathbb{C}[x, t]$ is a polynomial with $f(x, t)$ having a multiple root in x for almost all t, then f is *not* irreducible.

Exercample 7. The ideal of 2-minors of $\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$ defines an irreducible variety in a reduced way.

Exercample 8. $\mathcal{X}_{d,r}$ is an irreducible variety of dimension $2r$. Its defining ideal is generated by the $(r+1)$-minors of $H_r(y)$. We have $\mathcal{X}_{d,r}^{\text{sing}} = \mathcal{X}_{d,r-1}$.

Exercample 9.

- How does Thom–Porteous extend Bézout's theorem?
- What is the degree of $\mathbb{P}(\mathcal{X}_{d,r}) \subseteq \mathbb{P}^d$?