Lecture 4: Thom–Porteous formula and engineering applications

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Exercample 1. Show that if M is a matrix of $e \times f$ indeterminates (so $n+1 = ef$), then $D_r(M) \subseteq \mathbb{P}^{ef-1}$ is an irreducible variety of expected dimension.

Exercample 2. Compute the number of intersection points in $\mathbb{P}^2 \times \mathbb{P}^1$ of (transverally intersecting) hypersurfaces of bi-degrees $(2,0), (2,6), (6,7)$.

Exercample 3. Assume M has linear entries (for example ef distinct interminates). Show that $\deg D_{e-1}(M) = \binom{f}{e-1}$ by using the Kernel incidence

$$\mathcal{K} = \left\{ \, (x, [v]) \in \mathbb{P}^n \times \mathbb{P}^{e-1} \;\middle|\; M(x) \cdot v = 0 \, \right\}.$$

Check that this agrees with the theorem below.

Exercample 4. Evaluate this in the case $e = 3,\ f = 3,\ r = 1$.

Exercample 5. If $H = \frac{d}{c}$, $\hat{H} = \frac{b}{a}$, derive from this the *Walsh polynomial system*: $\hat{H}$ is a critical point if and only if there exists $g \in \mathbb{R}[z]_{\leq N-n-1}$ such that

$$a \cdot d - b \cdot c = \tilde{a}^2 \cdot g.$$

Exercample 6. Show that for almost all (c,d), the critical points (a,b) satisfy that a has n distinct roots and $\gcd(a,b) = 1$. If we normalize $a_n = 1$, then the set of critical points is finite.

Exercample 7. Show that if c has N distinct roots, then for *all* $\tilde{a} \neq 0$ the following linear system has no nonzero solution:

$$\begin{bmatrix} c & \tilde{a}^2 \end{bmatrix} \cdot \begin{pmatrix} b \\ g \end{pmatrix} = 0.$$

Exercample 8. Derive this from Thom–Porteous formula! The matrix M has column degrees

$$\big[\underset{1}{1} \quad \underset{n}{0 \dots 0} \quad \underset{N-n}{2 \dots 2}\big].$$

Exercample 9.

- Derive a formula for the number of points $a \in \mathbb{P}^n$ where $M(a)$ is not of full rank!
- Study this in the case $n = 1$, $N = 3$ (distance to cone $\mathbb{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{A}^3$).