

# Lecture 1: Affine varieties

## Algebraic Geometry Tools for Polynomial Systems in Engineering

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FOR MATHEMATICS  
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# A rather unusual format for an algebraic geometry course

- ▷ Hybrid between blackboard and slides
- ▷ Interactive format: **Exercamples** on every slide!
- ▷ Very few proofs, but (hopefully) lots of intuition
- ▷ **Please** don't wait with questions until the end!
- ▷ Suggested introductory resources:
  - Cox, David A., John Little, and Donal O'Shea, *Ideals, Varieties, and Algorithms*
  - Clader, Emily, and Dustin Ross, *Beginning in Algebraic Geometry*
  - Sommese, Andrew J., and Charles W. Wampler, *The Numerical Solution of Systems of Polynomials*

# In the beginning, there was $\mathbb{V}$

Let  $R := \mathbb{C}[X_1, \dots, X_n]$  and  $\mathbb{A}^n := \mathbb{C}^n$ .

## Definition (Affine variety)

Let  $\mathcal{F} \subseteq R$  be a set of polynomials. The **affine variety** defined by  $\mathcal{F}$  is

$$X = \mathbb{V}(\mathcal{F}) := \{ x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in \mathcal{F} \}$$

▷ **Examples:** Linear subspaces, plane curves, hypersurfaces, determinantal varieties, ...

## Exercample

- ▷ Show that  $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle_R)$ , where  $\langle \mathcal{F} \rangle_R := \{ \sum_i g_i f_i \mid g_i \in R, f_i \in \mathcal{F} \}$
- ▷ If  $X, Y \subseteq \mathbb{A}^n$  are varieties, then so are  $X \cap Y$  and  $X \cup Y$ .

### Definition (Vanishing ideal)

Let  $X \subseteq \mathbb{A}^n$  be a subset. The *vanishing ideal* of  $X$  is

$$I(X) = \{ f \in R \mid f(x) = 0 \text{ for all } x \in X \}$$

### Exercample

Show that for all  $\mathcal{F}, X$

- ▷  $\mathbb{V}(\dots)$  and  $I(\dots)$  are *antitone*:  $\mathcal{F} \subseteq \mathcal{G} \implies \mathbb{V}(\mathcal{F}) \supseteq \mathbb{V}(\mathcal{G})$
- ▷  $\mathcal{F} \subseteq I(\mathbb{V}(\mathcal{F}))$     and     $I(\mathbb{V}(I(X))) = I(X)$
- ▷  $X \subseteq \mathbb{V}(I(X))$     and     $\mathbb{V}(I(\mathbb{V}(\mathcal{F}))) = \mathbb{V}(\mathcal{F})$
- ▷ What is  $I(p)$ ,  $p = (p_1, \dots, p_n) \in \mathbb{A}^n$ ?

# A quite unusual topology

## Definition (Zariski topology: closed, open, irreducible, dense)

Let  $\emptyset \neq X \subseteq \mathbb{A}^n$  be a subset.

- ▷  $A \subseteq X$  is *closed in  $X$*  if it is of the form  $\mathbb{V}(\mathcal{F}) \cap X$  for some  $\mathcal{F}$
- ▷  $O \subseteq X$  is *open in  $X$*  if  $O = X \setminus A$  for some closed  $A \subset X$
- ▷  $X$  is *irreducible* if whenever  $X = A_1 \cup A_2$  for closed  $A_i$ , then  $A_1 = X$  or  $A_2 = X$
- ▷ The closure  $\overline{Y}$  of  $Y \subseteq X$   $\bigcap_{Y \subseteq A \subseteq X} A$ ;  $Y \subseteq X$  is *dense in  $X$*  if  $\overline{Y} = X$

## Exercample

- ▷ Every open set of  $X$  is a union of  $D_X(f) = \{x \in X \mid f(x) \neq 0\}$ ,  $f \in R$
- ▷  $Y$  is dense in  $X$  if and only if every non-empty open set of  $X$  intersects  $Y$ .
- ▷  $X$  is irreducible iff every non-empty open set is dense.
- ▷ If  $X$  is a variety and  $Y \subseteq X$ , then  $Y$  is irreducible iff  $\overline{Y}$  is irreducible
- ▷  $X = \bigcup_{i=1}^c X_i$  uniquely for  $X_i$  closed irreducible,  $X_i \not\subseteq X_j$  for  $i \neq j$  (“components”)

# Hilbert's theorem about the zeroes

## Definition (Radical ideal, prime ideal, maximal ideal)

Let  $I \subseteq R$  be an ideal.  $I$  is a *radical ideal* if it equals its *radical*

$$\sqrt{I} = \{ f \in R \mid f^m \in I \text{ for some } m > 0 \} \supseteq I.$$

$I \subsetneq R$  is *prime* if  $fg \in I$  implies  $f \in I$  or  $g \in I$ .

$I$  is *maximal* if it is inclusion-maximal among ideals  $\subsetneq R$ .

## Exercample

$\sqrt{I}$  is an ideal.  $I(X)$  is a radical ideal. Maximal  $\implies$  prime  $\implies$  radical.

## Theorem (Hilbert Nullstellensatz)

$$I(\mathbb{V}(\mathcal{F})) = \sqrt{\langle \mathcal{F} \rangle_R}, \quad \mathbb{V}(I(X)) = \overline{X} \quad \forall \mathcal{F}, X$$

$$\{\text{radical, prime, maximal ideals}\} \xleftrightarrow[I]{\mathbb{V}} \{\text{varieties, irreducible varieties, points}\}$$

# Nullstellensatz in action

**Hypersurfaces:** If  $X = \mathbb{V}(f)$  with  $f = f_1^{e_1} \cdots f_m^{e_m}$ ,  $f_i$  irreducible polynomials, then

$$X = \mathbb{V}(f_1) \cup \cdots \cup \mathbb{V}(f_m), \quad I(X) = \langle f_1 \cdots f_m \rangle_R$$

## Exercample

Let  $I \subseteq R$  be an ideal such that  $\mathbb{V}(I) = \{\mathbf{0}\}$  and let  $g \in R$ .

- ▷ Show that  $\ell := \dim_{\mathbb{C}} R/I < \infty$ .
- ▷ Show that the linear map  $\cdot g: R/I \rightarrow R/I$  has characteristic polynomial  $(t - g(\mathbf{0}))^\ell$ .

## Exercample

Describe the vanishing ideal of the following variety of skew-symmetric matrices:

$$\left\{ A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \mid \text{rank } A < 4 \right\} \subseteq \mathbb{A}^6.$$

# Morphism is such a fancy word

## Definition (Morphisms of affine varieties, dominance, fiber)

Let  $X \subseteq \mathbb{A}^n$ ,  $Y \subseteq \mathbb{A}^m$  be varieties and  $\phi: X \rightarrow Y$ .

- ▷  $\phi$  is a *morphism* of affine varieties if it is given by polynomials: There exist polynomials  $\phi_1, \dots, \phi_m \in R = \mathbb{C}[x_1, \dots, x_n]$  with  $\phi = (\phi_1, \dots, \phi_m)|_X$ .
- ▷ A  $\phi$  is *dominant* if  $\overline{\phi(X)} = Y$ .
- ▷ The *fiber*  $\phi^{-1}(y)$  of  $y \in Y$  is the subset  $\{x \in X \mid \phi(x) = y\}$ .

## Exercample

- ▷ Morphisms  $X \rightarrow \mathbb{A}^1 = \mathbb{C}$  are in bijection with  $R/I(X)$
- ▷ Show that a morphism  $X \rightarrow Y$  is **continuous**: Preimages of open/closed sets from  $Y$  are open/closed in  $X$
- ▷ Describe the image of  $\phi: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ ,  $(x, y) \mapsto (x, xy)$



## Definition (Coordinate ring)

The *coordinate ring* of  $X \subseteq \mathbb{A}^n$  is the  $\mathbb{C}$ -algebra (ring)  $\mathcal{O}(X) = R/I(X)$ .

## Exercample

- ▷ If  $X, Y \subseteq \mathbb{A}^n$  are such that  $X \cap Y = \emptyset$ , show that  $\mathcal{O}(X \cup Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$ . (Hint: Chinese remainder theorem).
- ▷ What is the coordinate ring of a set of points?
- ▷ Show that a morphism  $\phi: X \rightarrow Y$  induces a  $\mathbb{C}$ -algebra homomorphism  $\phi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ . Is this correspondence reversible?
- ▷ Show that  $X = \mathbb{V}(xy - 1)$  is not **isomorphic** to  $\mathbb{A}^1$  (no mutually-inverse morphisms).

# Dimensional analysis

## Definition (Dimension)

The *dimension* of a subset  $X \subseteq \mathbb{A}^n$  is

$$\dim X := \max \{ d \in \mathbb{N} \mid \exists Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d \subseteq X, Y_i \text{ irreducible closed subset} \}.$$

By convention,  $\dim \emptyset := -1$ .

For  $x \in X$ ,  $\dim_x X := \max\{\text{the same except } Y_0 \stackrel{!}{=} x\}.$

## Exercample

- ▶ Let  $X \subseteq \mathbb{A}^n$  be a variety, and  $X = X_1 \cup \cdots \cup X_r$  for subvarieties  $X_i$ . Show that  $\dim X = \max\{\dim X_1, \dots, \dim X_r\}$
- ▶ Find a variety  $X$  with a points  $x \in X$  with  $\dim X \neq \dim_x X$
- ▶ If  $X \rightarrow Y$  is surjective, show that  $\dim X \geq \dim Y$ .
- ▶ Show  $\dim \mathbb{A}^1 = 1$ . Can you argue  $\dim \mathbb{A}^2 = 2$ ?  $\dim \mathbb{A}^3 = 3$ ?

# The multiverse of dimension theory

## Theorem

*The following numbers all uniquely describe  $d = \dim X$ :*

- 1. The transcendence degree  $\text{trdeg}_{\mathbb{C}} \mathcal{O}(X)$ : The largest number of algebraically independent  $f_1, \dots, f_d \in \mathcal{O}(X)$ .*
- 2. The number  $d$  such that there exists a surjective morphism  $X \rightarrow \mathbb{A}^d$  with finite fibers.*
- 3. The number  $d$  such that a random selection of hyperplanes  $H_1, \dots, H_d \subseteq \mathbb{A}^n$  satisfies that  $X \cap H_1 \cap \dots \cap H_d$  is finite and non-empty.*
- 4. The largest number  $d$  such that there exist  $x_{i_1}, \dots, x_{i_d}$  with  $I(X) \cap \mathbb{C}[x_{i_1}, \dots, x_{i_d}] = \{0\}$ .*
- 5. If  $X$  is irreducible: The length of **every** non-extendable chain of irreducible subvarieties  $Z_0 \subsetneq \dots \subsetneq Z_d = X$ .*

# The powerhouse of algebraic geometry in applications

## Theorem (Fiber dimension theorem)

*Let  $\phi: X \rightarrow Y$  be a dominant morphism of irreducible varieties. Let  $y = \phi(x)$ , then*

$$\dim \phi^{-1}(y) \geq \dim X - \dim Y.$$

*Moreover, there is a dense open subset  $U \subseteq Y$  such that equality holds for all  $y$  in  $U$*

## Exercample

- ▷ Which result in linear algebra is the fiber dimension theorem generalizing?
- ▷ Find an example where equality does not always hold for all  $y \in Y$ .
- ▷ What is the dimension of the variety of  $m \times n$ -matrices of rank  $\leq r$ .

# How to be radical i

## Definition (Smoothness)

Let  $\langle f_1, \dots, f_s \rangle$  be the vanishing ideal of  $X \subseteq \mathbb{A}^n$ , then the smooth locus of  $X$  is

$$X^{\text{sm}} := \left\{ p \in X \mid \text{rank} \left[ \frac{\partial f_i}{\partial x_j}(p) \right]_{i,j} \geq n - \dim_p X \right\}, \quad X^{\text{sing}} = X \setminus X^{\text{sm}}$$

- ▷ The smooth locus of  $X$  is a dense open subset
- ▷ If  $X \subseteq Y$ , then  $X^{\text{sing}} \subseteq Y^{\text{sing}}$ . Points on multiple components of  $X$  are always singular
- ▷ **Useful:** If  $\mathbb{V}(f_1, \dots, f_m) = X$  and *these* equations satisfy the property from the definition for all  $p \in X$ , then  $\langle f_1, \dots, f_s \rangle_R = I(X)$

## Exercample

- ▷ Show the first statement for  $X = \mathbb{V}(f)$
- ▷ Find the smooth locus of  $\mathbb{V}(y^2 - x^3)$

## How to be radical ii

- ▷ Let  $\mathcal{F} = \mathcal{F}'(y) \cup \mathcal{F}''(x, y) \subseteq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$
- ▷ Let  $X = \mathbb{V}(\mathcal{F}) \subseteq \mathbb{A}^{n+m}$ ,  $Y = \mathbb{V}(\mathcal{F}') \subseteq \mathbb{A}^m$
- ▷ Assume that the equations  $\mathcal{F}''$  are of degree  $\leq 1$  in  $x$  and that for all  $b \in Y$ , the affine-linear system  $\mathcal{F}''(x, b)$  has constant rank  $r$
- ▷ Then  $X \rightarrow Y$  is an **affine bundle of rank  $r$** ,  $X =$  “base space”,  $Y =$  “total space”

### Lemma (Affine bundles transfer niceness from base to total space)

*If  $Y$  is irreducible/reduced/smooth/of dimension  $d$ , then  $X$  is irreducible/reduced/smooth/of dimension  $d + (n - r)$ .*

### Exercample

Let  $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1, x_2, y_1, y_2]$ . Show that  $\mathbb{V}(\mathcal{F})$  is a smooth irreducible variety of dimension 2. What do points  $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$  “represent”?

**Questions? Let's have lunch!**