

Lecture 1: Affine varieties

Algebraic Geometry Tools for Polynomial Systems in Engineering

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A rather unusual format for an algebraic geometry course

- ▷ Hybrid between blackboard and slides
- ▷ Interactive format: **Exercamples** on every slide!
- ▷ Very few proofs, but (hopefully) lots of intuition
- ▷ **Please** don't wait with questions until the end!
- ▷ Suggested introductory resources:
 - Cox, David A., John Little, and Donal O'Shea, *Ideals, Varieties, and Algorithms*
 - Clader, Emily, and Dustin Ross, *Beginning in Algebraic Geometry*
 - Sommese, Andrew J., and Charles W. Wampler, *The Numerical Solution of Systems of Polynomials*

In the beginning, there was \mathbb{V}

Let $R := \mathbb{C}[X_1, \dots, X_n]$ and $\mathbb{A}^n := \mathbb{C}^n$.

Definition (Affine variety)

Let $\mathcal{F} \subseteq R$ be a set of polynomials. The **affine variety** defined by \mathcal{F} is

$$X = \mathbb{V}(\mathcal{F}) := \{ x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in \mathcal{F} \}$$

- ▷ **Examples:** Linear subspaces, plane curves, hypersurfaces, determinantal varieties, ...

Exercexample

- ▷ Show that $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle_R)$, where $\langle \mathcal{F} \rangle_R := \{ \sum_i g_i f_i \mid g_i \in R, f_i \in \mathcal{F} \}$
- ▷ If $X, Y \subseteq \mathbb{A}^n$ are varieties, then so are $X \cap Y$ and $X \cup Y$.

In an ideal world

Definition (Vanishing ideal)

Let $X \subseteq \mathbb{A}^n$ be a subset. The *vanishing ideal* of X is

$$I(X) = \{ f \in R \mid f(x) = 0 \text{ for all } x \in X \}$$

Exercexample

Show that for all \mathcal{F}, X

- ▷ $\mathbb{V}(\dots)$ and $I(\dots)$ are *antitone*: $\mathcal{F} \subseteq \mathcal{G} \implies \mathbb{V}(\mathcal{F}) \supseteq \mathbb{V}(\mathcal{G})$
- ▷ $\mathcal{F} \subseteq I(\mathbb{V}(\mathcal{F}))$ and $I(\mathbb{V}(I(X))) = I(X)$
- ▷ $X \subseteq \mathbb{V}(I(X))$ and $\mathbb{V}(I(\mathbb{V}(\mathcal{F}))) = \mathbb{V}(\mathcal{F})$
- ▷ What is $I(p)$, $p = (p_1, \dots, p_n) \in \mathbb{A}^n$?

A quite unusual topology

Definition (Zariski topology: closed, open, irreducible, dense)

Let $\emptyset \neq X \subseteq \mathbb{A}^n$ be a subset.

- ▷ $A \subseteq X$ is *closed in X* if it is of the form $\mathbb{V}(\mathcal{F}) \cap X$ for some \mathcal{F}
- ▷ $O \subseteq X$ is *open in X* if $O = X \setminus A$ for some closed $A \subset X$
- ▷ X is *irreducible* if whenever $X = A_1 \cup A_2$ for closed A_i , then $A_1 = X$ or $A_2 = X$
- ▷ The closure \overline{Y} of $Y \subseteq X$ $\bigcap_{Y \subseteq A \subseteq X} A$; $Y \subseteq X$ is *dense in X* if $\overline{Y} = X$

Exercexample

- ▷ Every open set of X is a union of $D_X(f) = \{x \in X \mid f(x) \neq 0\}$, $f \in R$
- ▷ Y is dense in X if and only if every non-empty open set of X intersects Y .
- ▷ X is irreducible iff every non-empty open set is dense.
- ▷ If X is a variety and $Y \subseteq X$, then Y is irreducible iff \overline{Y} is irreducible
- ▷ $X = \bigcup_{i=1}^c X_i$ uniquely for X_i closed irreducible, $X_i \not\subseteq X_j$ for $i \neq j$ ("components")

Hilbert's theorem about the zeroes

Definition (Radical ideal, prime ideal, maximal ideal)

Let $I \subseteq R$ be an ideal. I is a *radical ideal* if it equals its *radical*

$$\sqrt{I} = \{ f \in R \mid f^m \in I \text{ for some } m > 0 \} \supseteq I.$$

$I \subsetneq R$ is *prime* if $fg \in I$ implies $f \in I$ or $g \in I$.

I is *maximal* if it is inclusion-maximal among ideals $\subsetneq R$.

Exercexample

\sqrt{I} is an ideal. $I(X)$ is a radical ideal. Maximal \implies prime \implies radical.

Theorem (Hilbert Nullstellensatz)

$$I(\mathbb{V}(\mathcal{F})) = \sqrt{\langle \mathcal{F} \rangle_R}, \quad \mathbb{V}(I(X)) = \overline{X} \quad \forall \mathcal{F}, X$$

$$\{ \text{radical, prime, maximal ideals} \} \xleftarrow[\substack{\mathbb{V} \\ I}]{} \{ \text{varieties, irreducible varieties, points} \}$$

Nullstellensatz in action

Hypersurfaces: If $X = \mathbb{V}(f)$ with $f = f_1^{e_1} \cdots f_m^{e_m}$, f_i irreducible polynomials, then

$$X = \mathbb{V}(f_1) \cup \cdots \cup \mathbb{V}(f_m), \quad I(X) = \langle f_1 \cdots f_m \rangle_R$$

Exercexample

Let $I \subseteq R$ be an ideal such that $\mathbb{V}(I) = \{\mathbf{0}\}$ and let $g \in R$.

- ▷ Show that $\ell := \dim_{\mathbb{C}} R/I < \infty$.
- ▷ Show that the linear map $\cdot g: R/I \rightarrow R/I$ has characteristic polynomial $(t - g(\mathbf{0}))^\ell$.

Exercexample

Describe the vanishing ideal of the following variety of skew-symmetric matrices:

$$\left\{ A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \mid \text{rank } A < 4 \right\} \subseteq \mathbb{A}^6.$$

Morphism is such a fancy word

Definition (Morphisms of affine varieties, dominance, fiber)

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be varieties and $\phi: X \rightarrow Y$.

- ▷ ϕ is a *morphism* of affine varieties if it is given by polynomials: There exist polynomials $\phi_1, \dots, \phi_m \in R = \mathbb{C}[x_1, \dots, x_n]$ with $\phi = (\phi_1, \dots, \phi_m)|_X$.
- ▷ A ϕ is *dominant* if $\overline{\phi(X)} = Y$.
- ▷ The *fiber* $\phi^{-1}(y)$ of $y \in Y$ is the subset $\{x \in X \mid \phi(x) = y\}$.

Exercample

- ▷ Morphisms $X \rightarrow \mathbb{A}^1 = \mathbb{C}$ are in bijection with $R/I(X)$
- ▷ Show that a morphism $X \rightarrow Y$ is *continuous*: Preimages of open/closed sets from Y are open/closed in X
- ▷ Describe the image of $\phi: \mathbb{A}^2 \rightarrow \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$

Definition (Coordinate ring)

The *coordinate ring* of $X \subseteq \mathbb{A}^n$ is the \mathbb{C} -algebra (ring) $\mathcal{O}(X) = R/I(X)$.

Exercexample

- ▷ If $X, Y \subseteq \mathbb{A}^n$ are such that $X \cap Y = \emptyset$, show that $\mathcal{O}(X \cup Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$. (Hint: Chinese remainder theorem).
- ▷ What is the coordinate ring of a set of points?
- ▷ Show that a morphism $\phi: X \rightarrow Y$ induces a \mathbb{C} -algebra homomorphism $\phi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Is this correspondence reversible?
- ▷ Show that $X = \mathbb{V}(xy - 1)$ is not isomorphic to \mathbb{A}^1 (no mutually-inverse morphisms).

Dimensional analysis

Definition (Dimension)

The *dimension* of a subset $X \subseteq \mathbb{A}^n$ is

$$\dim X := \max \{ d \in \mathbb{N} \mid \exists Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d \subseteq X, Y_i \text{ irreducible closed subset} \}.$$

By convention, $\dim \emptyset := -1$.

For $x \in X$, $\dim_x X := \max \{ \text{the same except } Y_0 \stackrel{!}{=} x \}$.

Exercexample

- ▷ Let $X \subseteq \mathbb{A}^n$ be a variety, and $X = X_1 \cup \cdots \cup X_r$ for subvarieties X_i . Show that $\dim X = \max\{\dim X_1, \dots, \dim X_r\}$
- ▷ Find a variety X with a point $x \in X$ with $\dim X \neq \dim_x X$
- ▷ If $X \rightarrow Y$ is surjective, show that $\dim X \geq \dim Y$.
- ▷ Show $\dim \mathbb{A}^1 = 1$. Can you argue $\dim \mathbb{A}^2 = 2$? $\dim \mathbb{A}^3 = 3$?

The multiverse of dimension theory

Theorem

The following numbers all uniquely describe $d = \dim X$:

1. *The transcendence degree $\text{trdeg}_{\mathbb{C}} \mathcal{O}(X)$: The largest number of algebraically independent $f_1, \dots, f_d \in \mathcal{O}(X)$.*
2. *The number d such that there exists a surjective morphism $X \rightarrow \mathbb{A}^d$ with finite fibers.*
3. *The number d such that a random selection of hyperplanes $H_1, \dots, H_d \subseteq \mathbb{A}^n$ satisfies that $X \cap H_1 \cap \dots \cap H_d$ is finite and non-empty.*
4. *The largest number d such that there exist x_{i_1}, \dots, x_{i_d} with $I(X) \cap \mathbb{C}[x_{i_1}, \dots, x_{i_d}] = \{\mathbf{0}\}$.*
5. *If X is irreducible: The length of **every** non-extendable chain of irreducible subvarieties $Z_0 \subsetneq \dots \subsetneq Z_d = X$.*

Theorem (Fiber dimension theorem)

Let $\phi: X \rightarrow Y$ be a dominant morphism of irreducible varieties. Let $y = \phi(x)$, then

$$\dim \phi^{-1}(y) \geq \dim X - \dim Y.$$

Moreover, there is a dense open subset $U \subseteq Y$ such that equality holds for all y in U

Exercexample

- ▷ Which result in linear algebra is the fiber dimension theorem generalizing?
- ▷ Find an example where equality does not always hold for all $y \in Y$.
- ▷ What is the dimension of the variety of $m \times n$ -matrices of rank $\leq r$.

How to be radical i

Definition (Smoothness)

Let $\langle f_1, \dots, f_s \rangle$ be the vanishing ideal of $X \subseteq \mathbb{A}^n$, then the smooth locus of X is

$$X^{\text{sm}} := \left\{ p \in X \mid \text{rank} \left[\frac{\partial f_i}{\partial x_j}(p) \right]_{i,j} \geq n - \dim_p X \right\}, \quad X^{\text{sing}} = X \setminus X^{\text{sm}}$$

- ▷ The smooth locus of X is a dense open subset
- ▷ If $X \subseteq Y$, then $X^{\text{sing}} \subseteq Y^{\text{sing}}$. Points on multiple components of X are always singular
- ▷ **Useful:** If $\mathbb{V}(f_1, \dots, f_m) = X$ and *these* equations satisfy the property from the definition for all $p \in X$, then $\langle f_1, \dots, f_s \rangle_R = I(X)$

Exercexample

- ▷ Show the first statement for $X = \mathbb{V}(f)$
- ▷ Find the smooth locus of $\mathbb{V}(y^2 - x^3)$

How to be radical ii

- Let $\mathcal{F} = \mathcal{F}'(y) \cup \mathcal{F}''(x, y) \subseteq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$
- Let $X = \mathbb{V}(\mathcal{F}) \subseteq \mathbb{A}^{n+m}$, $Y = \mathbb{V}(\mathcal{F}') \subseteq \mathbb{A}^m$
- Assume that the equations \mathcal{F}'' are of degree ≤ 1 in x and that for all $b \in Y$, the affine-linear system $\mathcal{F}''(x, b)$ has constant rank r
- Then $X \rightarrow Y$ is an **affine bundle of rank r** , X = “base space”, Y = “total space”

Lemma (Affine bundles transfer niceness from base to total space)

If Y is irreducible/reduced/smooth/of dimension d , then X is irreducible/reduced/smooth/of dimension $d + (n - r)$.

Exercample

Let $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1, x_2, y_1, y_2]$. Show that $\mathbb{V}(\mathcal{F})$ is a smooth irreducible variety of dimension 2. What do points $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$ “represent”?

Questions? Let's have lunch!