Lecture 2: Projective varieties

Algebraic Geometry Tools for Polynomial Systems in Engineering

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Today's menu

0. Left-over from lecture 1: The fiber dimension theorem
1. Projective varieties:
   - Homogeneous polynomials, projective Nullstellensatz
   - Dimension and non-emptyness of intersections
2. Hilbert functions
   - Hilbert polynomial, dimension, degree
   - A proof idea of Bézout & BKK
3. Intersection multiplicity

The powerhouse of algebraic geometry in applications

Theorem (Fiber dimension theorem)

Let $\phi \colon X \to Y$ be a dominant morphism of irreducible varieties. Let $y = \phi(x)$, then

$$\dim \phi^{-1}(y) \geq \dim X - \dim Y.$$

Moreover, there is a dense open subset $U \subseteq Y$ such that equality holds for all y in U.

Exercample

- Which result in linear algebra is the fiber dimension theorem generalizing?
- Find an example where equality does not always hold for all $y \in Y$.
- For later: What is the dimension of the variety of $m \times n$-matrices of rank $\leq r$?

From affine to projective space

Last time $R = \mathbb{C}[x_1, \ldots, x_n]$, today additionally $S = \mathbb{C}[x_0, \ldots, x_n]$

Theorem (Bézout theorem (affine version))

Let $f_1, \ldots, f_n \in R$ be such that $X = \mathbb{V}(f_1, \ldots, f_n)$ is a finite set of points. Then $\#X \leq \deg(f_1) \cdots \deg(f_n)$. If the f_i are sufficiently general, then equality holds.

- Would *love* to always have equality, but can fail for two reasons:
  1. Points should be counted with multiplicity
  2. Missing points are "at infinity"
- Solution to 1.: Intersection multiplicity
- Partial solution to 2.: Compactify $\mathbb{A}^n$ to $\mathbb{P}^n = \mathbb{A}^n \cup H_\infty$
- Better (?) solution to 2.: Refine solution bound (BKK theorem)

Projective space

Definition (Projective space, homogeneous coordinates, projective variety)

Projective space is $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, $v \sim v'$, if $v' = \lambda v$ for some $\lambda \in \mathbb{C}^\times$.

$a \in \mathbb{P}^n$ is written in *homogeneous coordinates* $[a_0 : a_1 : \ldots : a_n]$; unique up to scaling.

A *projective variety* is $\mathbb{V}(\mathcal{F}) = \{ z \in \mathbb{P}^n \mid f(z) = 0 \; \forall f \in \mathcal{F} \text{ homogeneous} \}$, $\mathcal{F} \subseteq S$.

- Every projective variety $X \neq \emptyset$ has *affine cone* (convention $\widehat{\emptyset} = \emptyset$)

$$\widehat{X} = \{ p \in \mathbb{A}^{n+1} \mid [p] \in X \} \cup 0 = \mathbb{V}_{\mathbb{A}^{n+1}}(\mathcal{F})$$

- Analogous definitions: Closed, open, irreducible, dense
- Projective varieties $\leftrightarrow$ affine varieties invariant up to scaling (except $\mathbf{0}$)

Exercample

Show that any lines $L_1, L_2 \subseteq \mathbb{P}^2$ intersect ($L_i = \mathbb{V}(\ell_i)$, $0 \neq \ell_i \in S_1$).

Déja-vu: Homogeneous vanishing ideals

Definition (Homogeneous vanishing ideal)

An ideal $I \subseteq S$ is homogeneous if it is generated by homogeneous polynomials. The homogeneous vanishing ideal of $X \subseteq \mathbb{P}^n$ is $I(X) = \bigoplus_{d \geq 0} \{ f \in S_d \mid f(X) = 0 \} \subseteq S$.
$I \subseteq S$ is *irrelevant* if $\sqrt{I} := \mathfrak{m} = \langle x_0, \ldots, x_n \rangle_S$, equiv. $\mathbb{V}_{\mathbb{A}^{n+1}}(I) = \{\mathbf{0}\}$.

Exercample

- $I \subseteq S$ is homogeneous iff $f \in I$ implies $f_j \in I$ for its graded comp. $f = \sum_{j=0}^{\deg f} f_j$
- $I(X)$ (homogeneous vanishing ideal) $= I(\widehat{X})$ (ideal of cone)

Theorem (Projective Nullstellensatz)

For homogeneous $\mathcal{F} \subseteq S$ with $\langle \mathcal{F} \rangle_S$ not irrelevant and for $X \subseteq \mathbb{P}^n$,

$$I(\mathbb{V}(\mathcal{F})) = \sqrt{\langle \mathcal{F} \rangle_R} \qquad \mathbb{V}(I(X)) = \overline{X} \qquad \forall \mathcal{F}, X$$

$$\{\textit{homogeneous radical/prime ideals} \neq \mathfrak{m}\} \underset{I}{\overset{\mathbb{V}}{\longleftrightarrow}} \{\textit{projective varieties, irreducible varieties}\}$$

Intersections work better in projective space

Lemma (Krull's Hauptidealsatz, baby case)

Let $X \subseteq \mathbb{A}^n$ *be an irreducible variety and* $f \in R$*, then either*

1. $f|_X = 0$*, then* $X \cap \mathbb{V}(f) = X$*;*
2. $f|_X =$ *const* $\neq 0$*, then* $X \cap \mathbb{V}(f) = \emptyset$*;*
3. $f|_X$ *not constant, then* $\dim(X \cap \mathbb{V}(f)) = \dim X - 1$*.*

Theorem (Intersection dimension bound)

Let $X, Y \subseteq \mathbb{A}^n$ *or* $\mathbb{P}^n$ *be irreducible varieties of dimension* d, e*. Then every irreducible component of* $X \cap Y$ *has dimension* $\geq d + e - n$*.*

In the projective case, if $e + f \geq n$*, then* $X \cap Y \neq \emptyset$*.*

Exercample

Let $f_1, \ldots, f_n \in S$ be homogeneous. Show that $X = \mathbb{V}_{\mathbb{P}^n}(f_1, \ldots, f_n) \neq \emptyset$.

Show that if X is finite, then $\dim \mathbb{V}(f_1, \ldots, f_k) = n - k$ for $k = 1, \ldots, n$.

Show that this is generally *fails* in $\mathbb{A}^3$.

Charting new territory

- Projective space has affine charts $U_0, \ldots, U_n$ given by

$$U_i = \mathbb{P}^n \setminus \mathbb{V}(x_i) = \{\, a \in \mathbb{P}^n \mid a_i \neq 0 \,\} \leftrightarrow \{\, (p_0, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n) \mid p \,\} \cong \mathbb{A}^n$$

- The *dehomogenization* of $f(x_0, \ldots, x_n) \in S_d$ is $f^{\mathrm{a}}(1, x_1, \ldots, x_n) \in R$
- Similarly, the *homogenization* of $f = \sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_b^{\alpha_n} \in R$ is $\sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_b^{\alpha_n}$
- For ideals $I \subseteq S$, $J \subseteq R$ the ideals $I^{\mathrm{a}} \subseteq R$, $J^{\mathrm{h}} \subseteq S$ are defined element-wise

Theorem (Switching between affine and projective varieties)

The maps $\mathbb{A}^n \supseteq X \mapsto \overline{X} \subseteq \mathbb{P}^n$, $\quad \mathbb{P}^n \supseteq Y \mapsto Y \cap U_0 \subseteq U_0 \cong \mathbb{A}^n$ *induce a bijection between affine varieties and projective varieties none of whose components are contained in* $H_\infty := \mathbb{V}(x_0)$. *On ideals, this correspondence is via (de)homogenization.*

Exercample

Let $X = \{\, (t, t^2, t^3) \mid t \in \mathbb{C} \,\} \subseteq \mathbb{A}^3$. Show that $X = \mathbb{V}(x_1^2 - x_2, x_1^3 - x_3)$. We will see later that these equations generate $I = I(X) \subseteq R$. Describe $\overline{X} \subseteq \mathbb{P}^3$ and investigate I^{h}.

Coffee break discussion!

Exercample

- Let $\phi \colon X \to Y$ be a morphism of affine varieties. If X is irreducible and ϕ is surjective, then Y is surjective.
- Show that the set of matrices $X \subseteq \mathbb{C}^{m \times n}$ of rank $\leq r$ is an irreducible variety
- Compute the dimension of X.

Exercample

- Let $X = \{ (t, t^2, t^3) \mid t \in \mathbb{C} \} \subseteq \mathbb{A}^3$. Show that $X = \mathbb{V}(x_1^2 - x_2, x_1^3 - x_3)$. We will see later that these equations generate $I = I(X) \subseteq R$.
- Describe $\overline{X} \subseteq \mathbb{P}^3$. Can you find generators of $I(\overline{X}) = I^{\mathrm{h}}$ up to radical?
- Compute the number of intersection points of X with a general plane $H \subseteq \mathbb{P}^3$.

Hilbert function, Hilbert polynomial

Definition (Hilbert function)

Let $V = \bigoplus_{d \geq 0} V_d$ be a graded vector space. The *Hilbert function* of V is $\mathrm{hf}_V(t) = \dim_{\mathbb{C}} V_t$. The Hilbert function of a projective variety $X \subseteq \mathbb{P}^n$ is $\mathrm{hf}_X = \mathrm{hf}_{S/I(X)}$.

Definition (Degree of a projective variety)

The *degree* of a projective variety $X \subseteq \mathbb{P}^n$ is the number of points of the set $X \cap H_1 \cap \cdots \cap H_{\dim X}$ for general hyperplanes $H_i = \mathbb{V}(\ell_i)$.

Exercample

- Compute the Hilbert function of a hypersurface $\mathbb{V}(f) \subseteq \mathbb{P}^n$, $f \in S_d$
- Compute the Hilbert function for a set of 3 points in $\mathbb{P}^2$

The Hilbert polynomial knows dimension and degree

Lemma

For every variety X there exists a polynomial $P(t) \in \mathbb{Q}[t]$ and a $t_0 \in \mathbb{Z}$ such that $\mathrm{hf}_X(t) = P(t)$ for $t \geq t_0$. This is the Hilbert polynomial P_X of X.

Exercample

- What is the Hilbert polynomial of $\mathbb{P}^n$? Of a hypersurface?
- Show that if $X \cap Y = \emptyset$, then $P_{X \cup Y} = P_X + P_Y$, but equality may not hold for Hilbert functions.

Theorem (Bézout's theorem, Hilbert polynomial form)

Let $\delta = \dim X$, then $P_X(t) = \dfrac{\deg(X)}{\delta!} t^\delta + O(t^{\delta-1})$.
If X is irreducible and $f \in S_d$ intersects X transversally (to be specified), then

$$P_{X \cap \mathbb{V}(f)}(t) = P_X(t) - P_X(t-d), \qquad \deg(X \cap \mathbb{V}(f)) = \deg(X) \cdot \deg(f).$$

The BKK theorem

- Let $A \subseteq \mathbb{N}^n$ be a set of supports (exponent vectors) and let

$$\mathbb{C}^A = \{\, f \in R \mid f = \sum_{\alpha \in A} f_\alpha x^\alpha \,\}.$$

- Let $\mathcal{P} := \mathrm{Conv}(A)$ be the convex hull, it is the Newton polytope of $f \in \mathbb{C}^A$
- Denote by $\mathbb{T}^n = (\mathbb{C} \setminus 0)^n = \mathbb{A}^n \setminus \mathbb{V}(x_1 \cdots x_n)$ the algebraic torus

Theorem (Bernstein–Khovanskii–Kushnirenko)

Let $f_1, \ldots, f_n \in \mathbb{C}^A$ be polynomials and let $X = \mathbb{V}(f_1, \ldots, f_n) \cap \mathbb{T}^n$. The number of isolated points in X is $\leq n!\,\mathrm{Vol}(\mathcal{P})$. If the f_i are general, $\#X = n!\,\mathrm{Vol}(\mathcal{P})$.

- Generalizes to $f_i \in \mathbb{C}^{A_i}$ with different supports $\rightsquigarrow$ mixed volume $n!\,\mathrm{mVol}(\mathcal{P}_1, \ldots, \mathcal{P}_n)$

Exercample

Explore the theorem for the BKK-general system
$\mathcal{F} = \{-1 + x - y + xy, 2 + x + \frac{1}{2}y - xy\}$. What are its roots in $\mathbb{A}^2$? In $\mathbb{P}^2$?

Multiplicity

Definition (Local ring, intersection multiplicity)

Let $p \in \mathbb{A}^n$. The *local ring* at p is $\mathcal{O}_{\mathbb{A}^n,p} = \left\{ \frac{f}{g} \in \mathbb{C}(x_1,\dots,x_n) \,\middle|\, g(p) \neq 0 \right\}$. The *multiplicity* of an ideal $I = \langle f_1,\dots,f_s\rangle \subseteq R$ at $p \in \mathbb{V}(I)$ is

$$\mathrm{mult}_p(I) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^n,p}/\langle f_1,\dots,f_s\rangle_{\mathcal{O}_{\mathbb{A}^n,p}}$$

- $0 < \mathrm{mult}_p(I) < \infty$ if and only if p is isolated in $\mathbb{V}(I)$ (a component)
- $\mathrm{mult}_p(I) = 1$ iff $\langle f_1,\dots,f_s\rangle_{\mathcal{O}_{\mathbb{A}^n,p}} = \langle x_1 - p_1,\dots,x_n - p_n\rangle_{\mathcal{O}_{\mathbb{A}^n,p}}$ if and only if the Jacobi matrix of $f_1,\dots,f_s$ has maximal rank at p

Exercample

- Show that this agrees with your experience from multiplicity of univariate polynomials
- Let $p = \mathbf{0} \in \mathbb{A}^3$. Compute $\mathrm{mult}_{\mathbf{0}}(x^3, y^3, z^3)$ and $\mathrm{mult}_{\mathbf{0}}(\text{all mon's of deg. } 3)$
- Let $f = y^2 - x^2(x+1)$ and $g = y$. Compute $\mathbb{V}(f,g)$ and the intersection multiplicities

Finally: Bézout with multiplicities

- The intersection multiplicity of $p \in \mathbb{P}^n$ at $I \subseteq S$ is defined by dehomogenizing to an affine chart $p \in U_i \subseteq \mathbb{P}^n$

Exercample

If $\mathbb{V}(I) \subseteq \mathbb{P}^n$ is a finite set $p_1, \ldots, p_r$, then $\mathrm{hf}_{S/I}(t)$ eventually stabilizes at the value $m = \sum_{i=1}^r \mathrm{mult}_{p_i}(I)$

Theorem (Projective Bézout with multiplicities)

If $f_1, \ldots, f_n \in S$ are homogeneous of degree d_i such that $X = \mathbb{V}(f_1, \ldots, f_n)$ is a finite set, then

$$\sum_{p \in X} \mathrm{mult}_p(I) = d_1 \cdots d_n.$$

Questions? Let's have lunch!