Lecture 4: Thom–Porteous formula and engineering applications

Algebraic Geometry Tools for Polynomial Systems in Engineering

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Today: Applying Thom–Porteous to the degree of Model Order Reduction

1. Determinantal varieties
    - The expected dimension
    - Thom–Porteous (in corank 1)
    - A glimpse at intersections in $\mathbb{P}^m \times \mathbb{P}^n$
2. Counting solutions to the MOR problem
    - Rational function approximation problem
    - The Walsh polynomial system
    - Irreducibiliy and reducedness of the Walsh variety
    - Counting the number of critical points
3. Outlook

Expected dimension of determinantal varieties

- In the following, let $M(x) \in S^{f\times e}$ be a matrix of homogeneous polynomials, $f \geq e$
- Assume the degrees of the entries are constant along columns (this can be weakened)

Definition (Degeneracy locus)

The r-th degeneracy locus of M is $D_r(M) = \{\, x \in \mathbb{P}^n \mid \operatorname{rank} M(x) \leq r \,\}$.

Lemma

We have $\dim D_r(M) \geq n - (e-r)(f-r)$ (unless this number is negative).
If equality holds, then $D_r(M)$ is of expected dimension.

Exercample

Show that if M is a matrix of $e \times f$ indeterminates (so $n+1 = ef$), then $D_r(M) \subseteq \mathbb{P}^{ef-1}$ is an irreducible variety of expected dimension.

Slight detour: Intersections in $\mathbb{P}^m \times \mathbb{P}^n$

- Let $T = \mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_n]$ and consider *bi-homogeneous polynomials* $f(x,y)$
- f defines bi-projective hypersurfaces $\mathbb{V}(f) = \{ (x,y) \in \mathbb{P}^m \times \mathbb{P}^n \mid f(x,y) = 0 \}$

Definition (Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$)

The *Chow ring* of $\mathbb{P}^m \times \mathbb{P}^n$ is $A^*(\mathbb{P}^m \times \mathbb{P}^n) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^{m+1}, \beta^{n+1} \rangle$. To a hypersurface $\mathbb{V}(f)$ with f of bi-degree d, e we associate the *class* $[\mathbb{V}(f)] := d\alpha + e\beta \in A$.

Theorem (Bézout/BKK theorem in $\mathbb{P}^m \times \mathbb{P}^n$)

Let $f_1, \dots, f_{m+n}$ be bi-homogeneous polynomials and $Z_i = [\mathbb{V}(f_i)]$ the associated classes. If $\mathbb{V}(f_1, \dots, f_{m+n}) \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is a finite set, then its size (counted with multiplicities) is the coefficient δ of $Z_1 \cdots Z_{m+n} = \delta \cdot \alpha^m \beta^n \in A$.

Exercample

Compute the number of intersection points in $\mathbb{P}^2 \times \mathbb{P}^1$ of (transverally intersecting) hypersurfaces of bi-degrees $(2,0), (2,6), (6,7)$.

Thom–Porteous, corank 1

Assume that $D_{e-1}(M)$ is of expected dimension $n-(f-e+1)$

Exercample

Assume M has linear entries (for example ef distinct interminates). Show that $\deg D_{e-1}(M) = \binom{f}{e-1}$ by using the Kernel incidence

$$\mathcal{K} = \left\{ (x,[v]) \in \mathbb{P}^n \times \mathbb{P}^{e-1} \;\middle|\; M(x)\cdot v = 0 \right\}.$$

Check that this agrees with the theorem below.

- Consider the rational function $\Psi(T) = \frac{1}{(1-d_1T)\cdots(1-d_eT)}$
- Let $\{\Psi\}^k$ be the k-th coefficient in the series expansion $\Psi = \sum_{k\geq 0}\{\Psi\}^k T^k$

Theorem (Giambelli–Thom–Porteous, corank 1)

$$\deg X = \{\Psi\}^{f-e+1}.$$

Thom–Porteous, any rank

- $\{\Psi\}^k$ is the k-th coefficient in the expansion $\Psi = \frac{1}{(1-d_1T)\cdots(1-d_eT)} = \sum_{k\geq 0}\{\Psi\}^k T^k$
- Form the Toeplitz-like matrix

$$\mathbb{D}_b^a = \begin{bmatrix} \{\Psi\}^b & \{\Psi\}^{b+1} & \cdots & \{\Psi\}^{b+a-1} \\ \{\Psi\}^{b-1} & \{\Psi\}^b & \cdots & \{\Psi\}^{b+a-2} \\ \vdots & \vdots & \ddots & \vdots \\ \{\Psi\}^{b-a+1} & \{\Psi\}^{b-a+2} & \cdots & \{\Psi\}^b \end{bmatrix} \in \mathbb{Z}^{a\times a}$$

Theorem (Thom–Porteous)

If $\dim D_r(M) = n - (e-r)(f-r)$, *then* $\deg D_r(M) = \det \mathbb{D}_{f-r}^{e-r}$.

Exercample

Evaluate this in the case $e = 3$, $f = 3$, $r = 1$.

- Variants for $e > f$, for suitable mixed degrees, over other varieties than $\mathbb{P}^n$, ...

Model Order Reduction

- A (stable discrete-time SISO LTI ...) model is represented by its impulse response $h = (h_1, h_2, \dots) \in \ell^2$, or equivalently its transfer function

$$H(z) = \frac{d(z)}{c(z)} = \sum_{k \geq 1} h_k z^{-k}.$$

- Model is of order N if h obeys linear recurrence of order N, equivalently if $\deg c \leq N$, $\deg d < N$
- Norm/distance of model(s) is given by

$$||h||_{\ell^2} = ||H||_{\mathcal{H}_2} := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} |H(z)| \frac{\mathrm{d}z}{z}$$

- Model order reduction: Minimize $||h - \hat{h}||$ among $\hat{h}$ of order $\leq n$.

Critical equations

Theorem (Walsh's theorem on rational function approximation)

Assume $\hat{H}$ *has simple poles* $\omega_1, \ldots, \omega_n$. $\hat{H}$ *is a critical point to* $\hat{H} \mapsto ||H - \hat{H}||^2$ *if and only if*

$$\hat{H}(\omega^{-1}) = H(\omega^{-1}), \quad \hat{H}'(\omega^{-1}) = H'(\omega^{-1}), \qquad \forall \omega \in \{\omega_1, \ldots, \omega_n\}.$$

- For a polynomial $a(z) \in \mathbb{C}[z]_{\leq n}$ let $\tilde{a}(z) = z^n a(1/z) \in \mathbb{C}[z]_{\leq n}$.

Exercample

If $H = \frac{d}{c}$, $\hat{H} = \frac{b}{a}$, derive from this the *Walsh polynomial system*: $\hat{H}$ is a critical point if and only if there exists $g \in \mathbb{R}[z]_{\leq N-n-1}$ such that

$$a \cdot d - b \cdot c = \tilde{a}^2 \cdot g.$$

The Walsh variety

Compact notation:

$a \in A = \mathbb{C}[z]_{\le n}$, $b \in B = \mathbb{C}[z]_{<n}$, $c \in C = \mathbb{C}[z]_{\le N}$, $d \in D = \mathbb{C}[z]_{<n}$, $g \in G = \mathbb{C}[z]_{\le N-n-1}$

Definition (Walsh variety)

The *Walsh variety* is

$$\mathcal{W} = \{ (a,b,c,d,g) \mid a \cdot d - b \cdot c = \tilde{a}^2 \cdot g \} \subseteq (A \setminus 0) \times B \times C \times D \times G.$$

Theorem (Kayser–Lagauw 2026+)

$\mathcal{W}$ is an irreducible and reduced complete intersection $(A \setminus 0) \times B \times C \times D \times G$ of dimension $2N+2$.

Exercample

Show that for almost all (c,d), the critical points (a,b) satisfy that a has n distinct roots and $\gcd(a,b) = 1$. If we normalize $a_n = 1$, then the set of critical points is finite.

The Walsh MEP has no “bad locus”

- The Walsh polynomial system can be written as

$$\underbrace{\begin{bmatrix} ad & c & \tilde{a}^2 \end{bmatrix}}_{=:M(a;c,d)} \cdot \begin{pmatrix} 1 \\ b \\ g \end{pmatrix} = 0, \qquad M(a;c,d) \in \mathbb{C}[a_0, \ldots, a_n]^{(N-n)\times(N+1)}.$$

- Here $v = (1; b; g)^{\mathsf{T}}$ is the (unknown) kernel vector of size $1 + n + (N - n)$
- To pass to determinantal variety $D_N(M)$ need to “homogenize” v to allow $1 = 0$

Exercample

Show that if c has N distinct roots, then for *all* $\tilde{a} \neq 0$ the following linear system has no nonzero solution:

$$\begin{bmatrix} c & \tilde{a}^2 \end{bmatrix} \cdot \begin{pmatrix} b \\ g \end{pmatrix} = 0.$$

The algebraic degree of Model Order Reduction

Theorem

If c has N distinct roots and d is general, then the number of complex solutions to the Walsh polynomial system is

$$\sum_{j=0}^{n} \binom{N-n-1+j}{j} 2^j.$$

Exercample

Derive this from Thom–Porteous formula! The matrix M has column degrees

$$\big[\underbrace{1}_{1} \quad \underbrace{0 \ldots 0}_{n} \quad \underbrace{2 \ldots 2}_{N-n}\big].$$

System identification: A similar story

- System identification is the task to find to a given datapoint $y \in \mathbb{R}^N$ the closest point

$$\hat{y} \in \mathcal{X}_{N-1,n} = \left\{ \hat{y} \in \mathbb{A}^N \;\middle|\; \operatorname{rank} \begin{bmatrix} \hat{y}_0 & \hat{y}_1 & \cdots & \hat{y}_r \\ \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{N-r-1} & \hat{y}_{N-r} & \cdots & y_{N-1} \end{bmatrix} \leq r \right\}$$

- The problem of system identification in the 2-norm case also leads to a MEP
- Here the homogenization step is *non-trivial* and can cause difficulties!
- Polynomial matrix $M(a)$ has column degrees $\begin{bmatrix} \underbrace{1}_{1} & \underbrace{3 \ldots 3}_{N-2n} \end{bmatrix}$

Exercample

- Derive a formula for the number of points $a \in \mathbb{P}^n$ where $M(a)$ is not of full rank!
- Study this in the case $n = 1$, $N = 3$ (distance to cone $\mathbb{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{A}^3$).