Affine and projective varieties

GIT & NaHC Reading Seminar

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IN THE SCIENCES

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Background: Spaces with functions

Definition (Space with functions)

Let $\mathbb{K} = \mathbb{C}$ (or any field). A **space with functions** is a top. space X together with a sheaf of functions $\mathcal{O}$, that is for each open U a $\mathbb{K}$-subalgebra $\mathcal{O}(U) \subseteq \mathrm{Maps}(U, \mathbb{K})$, such that

1. $\mathcal{O}$ is closed under restriction: If $V \subseteq U$ and $f \in \mathcal{O}(U)$, then $f|_V \in \mathcal{O}(V)$
2. $\mathcal{O}$ is closed under gluing: Given $U = \bigcup_i U_i$, $f_i \in \mathcal{O}(U_i)$ agreeing on overlaps, then there's a (unique) $f \in \mathcal{O}(U)$ with $f_i = f|_{U_i}$.

A **morphism** of spaces with functions $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ pulling back functions: For all $U \subseteq Y$ and $g \in \mathcal{O}_Y(U)$, $g \circ f \in \mathcal{O}_X(f^{-1}U)$.

- C^α-manifolds, $\alpha \in \mathbb{N}_0 \cup \{\infty, \omega\}$, are spaces with functions over $\mathbb{R}$!
- Complex manifolds and (reduced) complex analytic spaces are spaces w.fun. over $\mathbb{C}$!
- Smooth/holomorphic maps are precisely morphisms of spaces with functions!

How we'll define algebraic varieties (informally)

- Manifolds can be defined as spaces with functions with two properties
  - Locally they are isomorphic to open sets of $\mathbb{R}^n$ ($\mathbb{C}^n$) with C^α (holom.) functions, in fact such open sets are *exactly* those which can be used for charts
  - The topology is nice: Second-countable and Hausdorff
- Algebraic varieties are defined as spaces with functions over an alg. closed $\mathbb{K}$ such that
  - Locally they are isomorphic to affine varieties, these are closed (!) subsets of $\mathbb{K}^n$ defined by polynomial equations, with topology and functions determined by polynomials
  - They are quasi-compact (compact w/o Hausdorff) and "separated" (?)

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The Zariski topology

Let $\mathbb{C}[\underline{T}] := \mathbb{C}[T_1, \ldots, T_n]$ be the polynomial ring in n variables

Definition (Vanishing set, vanishing ideal, Zariski-closed)

1. The **vanishing set** of $\mathcal{F} \subseteq \mathbb{C}[\underline{T}]$ is $\mathrm{V}(\mathcal{F}) := \{\, x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in \mathcal{F} \,\}$; such sets are called **Zariski-closed**.
2. The **vanishing ideal** of $X \subseteq \mathbb{C}^n$ is $I(X) := \{\, f \in \mathbb{C}[\underline{T}] \mid f(x) = 0 \ \forall x \in X \,\}$.

- $\mathrm{V}(\mathcal{F}) = \mathrm{V}(\langle \mathcal{F} \rangle_{\mathbb{C}[\underline{T}]})$, where $\langle \mathcal{F} \rangle_{\mathbb{C}[\underline{T}]}$ is the ideal generated by $\mathcal{F}$
- $\bigcap_i \mathrm{V}(\mathfrak{a}_i) = \mathrm{V}(\bigcup_i \mathfrak{a}_i) = \mathrm{V}(\sum_i \mathfrak{a}_i)$ and $\mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b}) = \mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{V}(\mathfrak{a} \cdot \mathfrak{b})$
- Distinguished open sets $\mathrm{D}(f) := \mathbb{C}^n \setminus \mathrm{V}(f)$ form a basis of the Zariski-topology
- Similarly $I(\bigcup_i X_i) = \bigcap_i I(X_i)$ and $I(X \cap Y) \supseteq I(X) + I(Y)$, generally not equal:

$$X = \{\, T_2 = 0 \,\}, \ Y = \{\, T_2 = T_1^2 \,\}, \qquad X \cap Y = P = \{(0,0)\}$$

$$I(X) = \langle T_2 \rangle_{\mathbb{C}[\underline{T}]}, \ I(Y) = \langle T_1^2 - T_2 \rangle, \quad I(X) + I(Y) = \langle T_1^2, T_2 \rangle \subsetneq I(P) = \langle T_1, T_2 \rangle$$

Hilbert's Nullstellensatz: I and V are inverse to each other!

- $\mathrm{V}(I(X)) = \overline{X}$ (in the Zariski topology)
- **Nullstellensatz:** $I(\mathrm{V}(\mathcal{F})) = \sqrt{\langle \mathcal{F} \rangle}$, where $\sqrt{\mathfrak{a}} := \{\, f \in \mathbb{C}[\underline{T}] \mid f^k \in \mathfrak{a} \text{ for some } k \,\}$
- A top. space is **irreducible** if it is not the union of two proper closed subsets

$$
\begin{array}{ccc}
\{\text{ closed subsets } V \subseteq \mathbb{C}^n \} & \underset{\mathrm{V}(-)}{\overset{I(-)}{\rightleftarrows}} & \{\text{ rad. ideals } \mathfrak{a} \subseteq \mathbb{C}[\underline{T}] \} \\
\cup & & \cup \\
\{\text{ irreducible closed subsets } X \} & \longleftrightarrow & \{\text{ prime ideals } \mathfrak{p} \} \\
\cup & & \cup \\
\{\text{ points } p \} & \xleftrightarrow{\mathfrak{m}_p = \langle T_1 - p_1, \ldots, T_n - p_n \rangle} & \{\text{ maximal ideals } \mathfrak{m} \}
\end{array}
$$

Coordinate rings and polynomial functions on open sets

Definition (Coordinate ring, rational functions, regular functions, $\mathcal{O}_X$)

1. The **coordinate ring** of a closed subset $X \subseteq \mathbb{C}^n$ is $A(X) := \mathbb{C}[\underline{T}]/I(X)$.
2. If X is irreducible, then $K(X) := \mathrm{Frac}(A(X))$ is its field of **rational functions**.
3. $f \in K(X)$ is **regular** in $p \in X$ if $\exists g, h \in A(X)$ with $f = \frac{g}{h}$ and $h(p) \neq 0$.
   f is regular on $U \subseteq X$ if it is regular in each $p \in U$. Notation: $f \in \boldsymbol{\mathcal{O}_X}(U)$.

- X is irreducible if and only if $A(X)$ is an integral domain
- If $U = \mathrm{D}(f) := X \setminus \mathrm{V}(f)$, then $\mathcal{O}_X(U) = A(X)[f^{-1}]$, in particular $\mathcal{O}_X(X) = A(X)$
- If $X = X_1 \cup \cdots \cup X_r$, X_i irr. closed, then $f \colon U \to \mathbb{C}$ is regular iff the $f|_{U \cap X_i}$ are
- The Nullstellensatz holds word-for-word for subsets of X and ideals in $A(X)$!

Definition (Affine variety)

An **affine variety** is a space with functions isomorphic to $(X, \mathcal{O}_X)$, $X \subseteq \mathbb{C}^n$ closed.

Examples

- Affine space $\mathbb{A}^n := \mathbb{C}^n$ is an irreducible affine algebraic variety with $A(\mathbb{C}^n) = \mathbb{C}[\underline{T}]$
- Hypersurfaces $\mathrm{V}(f) \subseteq \mathbb{A}^n$ are irreducible iff f is irreducible (if f has no rep. factors)
- Familiar examples: (affine) linear spaces and plane conics
- $\mathrm{GL}(n, \mathbb{C}) = \{\, A \in \mathbb{C}^{n \times n} \mid \det A \neq 0 \,\} \subseteq \mathbb{A}^{n^2}$ with functions $\mathcal{O}_{\mathbb{A}^{n^2}}|_{\mathrm{GL}(n,\mathbb{C})}$ is an affine variety, in fact

$$\mathrm{GL}(n, \mathbb{C}) \cong \{\, (A, y) \in \mathbb{C}^{n^2+1} \mid \det A \cdot y = 1 \,\}$$

- Generally, for $X \subseteq \mathbb{C}^n$ affine, $f \in A(X)$ the open subspace $\mathrm{D}(f) \subseteq X$ is affine:

$$\mathrm{D}(f) \cong \{\, (x_1, \ldots, x_n, \tilde{x}) \in \mathbb{C}^{n+1} \mid x \in X,\ f(x) \cdot \tilde{x} = 1 \,\} = \mathrm{V}(\langle I(X), f(\underline{T}) \cdot \tilde{T} - 1 \rangle)$$

⇝ Varieties have a basis consisting of affine varieties (!)

- $\mathbb{P}^1$ (soon to be defined) and $\mathbb{A}^2 \setminus \{(0, 0)\}$ are *not* affine

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The all-knowing coordinate ring and $\mathrm{mSpec}(A)$

Definition (Maximal spectrum, mSpec)

For a ring A let $\mathrm{mSpec}(A)$ be the set of maximal ideals with the topology given by closed sets $\{\, \mathfrak{m} \in \mathrm{mSpec}(A) \mid \mathcal{F} \subseteq \mathfrak{m} \,\}$ for $\mathcal{F} \subseteq A$

- ▷ For affine X the Nullstellensatz describes a homeomorphism $X \leftrightarrow \mathrm{mSpec}(A(X))$
- ▷ For $p \leftrightarrow I(p) = \mathfrak{m}$ and $f \in A(X)$ we have $f(p) = f \mod \mathfrak{m} \in A(X)/\mathfrak{m} = \mathbb{C}$
- ▷ For integral domains A one can similarly construct a sheaf of functions on $\mathrm{mSpec}(A)$

$\rightsquigarrow$ In this way $X \cong \mathrm{mSpec}(A(X))$ as spaces with functions

The all-knowing coordinate ring and $\mathrm{mSpec}(A)$

- ▷ A morphism $f\colon X \to Y$ induces a $\mathbb{C}$-algebra hom. $f^*\colon A(Y) \to A(X)$, $f^*(g) = g \circ f$
- ▷ A ring hom. $\phi\colon A(Y) \to A(X)$ induces a map[1] $\phi^{-1}\colon \mathrm{mSpec}(A(X)) \to \mathrm{mSpec}(A(Y))$
- ▷ Identifying $Y = \mathrm{mSpec}(A(Y))$, $f\colon X \to Y$ defined by $f(p) = \phi^{-1}(\mathfrak{m}_p)$ is continuous and pulls regular functions back, hence f is a morphism

⇝ A morphism of affine varieties f is the same as a $\mathbb{C}$-alg.hom. of their coordinate rings

- ▷ Every finitely generated $\mathbb{C}$-algebra that is reduced (integral) arises as the coordinate ring of an (irreducible) variety (A is reduced if $a^k = 0$ implies $a = 0$.)

⇝ The category of (irreducible) affine varieties is anti-equivalent to the category of fin.gen. reduced (integral) $\mathbb{C}$-algebras!

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[1] This is true since $A(X)$ is a finitely generated $\mathbb{C}$-algebra. It is *not* true for all comm. rings, e.g. $\mathbb{Z} \subseteq \mathbb{Q}$.

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The Zariski topology on $\mathbb{P}^n$

- Let $S := \mathbb{C}[T_0, \ldots, T_n]$ be the polynomial ring in $n+1$ variables
- S has a standard grading $S = \bigoplus_{d \geq 0} S_d$, elements of $\bigcup_d S_d$ are homogeneous
- An ideal $\mathfrak{a} \subseteq S$ is graded if it is generated by homogeneous ideals
- $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1}) = (\mathbb{C}^{n+1} \setminus 0)/\sim$, $x \sim y$ iff $\mathbb{C}x = \mathbb{C}y$.

Definition (Vanishing set, graded vanishing ideal, Zariski-closed)

1. The **vanishing set** of homog. $\mathcal{F} \subseteq S$ is $\mathrm{V}_+(\mathcal{F}) := \{ x \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \in \mathcal{F} \}$; such sets are called **Zariski-closed**.
2. The **vanishing ideal** of $X \subseteq \mathbb{P}^n$ is $I(X) := \langle \{ f \in S \text{ homog.} \mid f(x) = 0 \ \forall x \in X \} \rangle_S$.

- **Projective Nullstellensatz:** $I(\mathrm{V}_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for $\mathfrak{a} \neq S_+ := \bigoplus_{d>0} S_d$ graded

⇝ $\{$ Closed sets $\} \leftrightarrow \{$ grad. rad. ideals $\neq S_+ \}$, $\{$ irr. $\} \leftrightarrow \{$ gr. primes $\subsetneq S_+ \}$, $p \leftrightarrow$?

Functions on projective varieties

Definition (Homogeneous coordinate ring, regular functions, $\mathcal{O}_X$)

1. The **homogeneous coordinate ring** of a closed subset $X \subseteq \mathbb{P}^n$ is $S(X) := S/I(X)$.
2. For $U \subseteq X$, $f\colon U \to \mathbb{C}$ is **regular** if it is locally (i.e. on a cover U_i) of the form $f(p) = \frac{g(p)}{h(p)}$ $\forall p \in U_i$, $g, h \in S(X)_d$, $h(p) \neq 0$. Notation again $f \in \boldsymbol{\mathcal{O}_X}(U)$.

- X is irreducible if and only if $S(X)$ is an integral domain
- If $U = \mathrm{D}_+(f) := X \setminus \mathrm{V}_+(f)$, $f \in S_d$, $d > 0$, then

$$\mathcal{O}_X(U) = S(X)[f^{-1}]_0 = \left\{ \frac{g}{f^k} \,\middle|\, k \geq 0,\ g \in S_{dk} \right\}$$

- One can show that $\mathcal{O}_X(X) = \mathbb{C}$ (constants) for $X \subseteq \mathbb{P}^n$ closed irreducible

Definition (Affine variety)

A **projective variety** is a space with functions isomorphic to $(X, \mathcal{O}_X)$, $X \subseteq \mathbb{P}^n$ closed.

Examples

- $\mathbb{P}^n$ is an irreducible projective variety, and so are hypersurfaces $\mathrm{V}_+(f)$
- $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere with open affine cover $\mathbb{C} \cup (\mathbb{C}^\times \cup \{\infty\})$
- (Smooth) cubic plane curves $\mathrm{V}_+(g) \subseteq \mathbb{P}^2$ a.k.a. elliptic curves are cool
- Open subsets of projective varieties $U \subseteq X$ are **quasi-projective varieties**, they are "never" projective varieties unless $U = X$
- Products (defined on next slide) of (quasi-)projective varieties are (quasi-)projective
- The homogeneous coordinate ring is not uniquely determined by X alone
  Example: The conic $\mathrm{V}_+(T_1^2 - T_0T_2) \subseteq \mathbb{P}^2$ is isomorphic to a line $\mathrm{V}_+(T_2)$, but $\mathbb{C}[T_0, T_1, T_2]/\langle T_1^2 - T_0T_2 \rangle \ncong \mathbb{C}[T_0, T_1]$

Products of varieties and (finally) the definition of a variety

- Let X, Y be spaces with functions, covered by finitely many affine varieties
- We want to turn the product $X \times Y$ into a space satisfying "the" universal property
- If $X \subseteq \mathbb{C}^n$, $Y \subseteq \mathbb{C}^m$, define $X \times Y$ to be the affine variety $X \times Y \subseteq \mathbb{C}^{m+n}$, one has $A(X \times Y) \cong A(X) \otimes_{\mathbb{C}} A(Y)$ as $\mathbb{C}$-algebras
- If $X = \bigcup_i U_i$, $Y = \bigcup_j V_i$ with U_i, V_j affine, then $X \times Y = \bigcup_{i,j} U_i \times V_j$, and we **define** the topology and functions in this way (!)

Definition

A variety is a space with functions which has a finite open cover of affine varieties, such that the subset $\{\, (x, x) \mid x \in X \,\} \subseteq X \times X$ is closed.

- Products and open sets of varieties are varieties
- Quasi-projective varieties are varieties!

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Relevant topics

- Dimension (!!!)
- Function germs and stalks of spaces with functions, tangent space, smooth points
- Proper varieties (main thm. of elimination theory)
- Algebraic vector bundles, line bundles, $\mathcal{O}_{\mathbb{P}^n}(n)$
- Complex varieties vs. complex manifolds (topology, Chow, GAGA, ...)
- Do morphisms of projective varieties come from graded ring homomorphisms?
- Are all varieties quasi-projective?
- Rational functions/maps on varieties
- Can we do all of this over $\mathbb{R}$?
- What are sheaves, schemes and locally ringed spaces?

Thank you! Questions?

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agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf, 2021.