GIT: Algebraic groups and Invariant theory

Barbara Betti

30 November 2023

1 Algebraic groups and G-varieties

We consider algebraic varieties over $\mathbb{C}$, not necessarily irreducible. The algebra of *regular functions on* X is denoted by $\mathbb{C}[X]$, that corresponds to the coordinate ring if X is affine. The field of rational functions is $\mathbb{C}(X)$.

Definition 1. An *algebraic group* is a variety G that has a compatible group structure, i.e. the multiplication and the inverse map

$$\begin{aligned} \cdot : G \times G &\longrightarrow G, \qquad & \iota : G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h & g &\longmapsto g^{-1} \end{aligned}$$

are morphism of varieties. We say that G is an *affine algebraic group* if G is an affine variety.

Definition 2. A *morphism* of algebraic groups G and H is a morphism of variety $f : G \to H$ that is also a group homomorphism. An *algebraic subgroup* of G is a closed subvariety H such that $H \hookrightarrow G$ is a morphism of algebraic groups. An *algebraic quotient* of G is an algebraic group G' such that there exists a morphism $f : G \to G'$ and surjective

The connected component G^0 containing the neutral element e_G is said *neutral component* and it is a closed normal subgroup of G such that the quotient G/G^0 is finite.

Proposition 3. *If $f : G \to H$ is a morphism of algebraic groups, then the image of f is a closed subgroup and, if f is injective, it is a closed immersion.*

Example 4.

1. Any *finite group* is algebraic.
2. The *general linear group* GL_n is an algebraic group. We already know that it is an affine variety. Since the coefficients of the product AB of two matrices are polynomial functions in the coefficients of A and B, we can say that the product map (as well the inverse map) is a morphism of varieties. The *special linear group* SL_n is also algebraic since it is a closed subgroup of GL_n, defined by $\Delta = 1$, where Δ is the determinant. We will see that any algebraic group is linear, i.e. it is a subgroup of GL_n defined by polynomial equations.
3. The multiplicative group $\mathbb{C}^* \simeq \mathrm{GL}_1$ and the additive group $\mathbb{C} \hookrightarrow \mathrm{GL}_2$, $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.
4. The $n-$dimensional torus $(\mathbb{C}^*)^n$, that is isomorphic to diagonal invertible matrices, is algebraic.
5. Let $U_n \subset \mathrm{GL}_n$ the subgroup of upper triangular matrices with 1 on the diagonal, this is a nilpotent group (i.e. $A - I_n$ is nilpotent for every $A \in U_n$). It is algebraic as it is a closed subgroup of GL_n.

6. Every *elliptic curve* in $\mathbb{P}^2$ is an non-affine algebraic group.

$\diamond$

Proposition 5. *An algebraic group G is a smooth variety and its components are the cosets (or lateral classes) $gG^0 :=\{gh | h \in G^0\}$, for $g \in G$.*

Definition 6. A *G-variety* is a variety X equipped with an action of the algebraic group G,

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

that is a morphism of varieties. We say that α is an *algebraic $G-$action*.

An algebraic action $\alpha : G \times X \longrightarrow X$ induces a linear action of G on the coordinate ring $\mathbb{C}[X]$:

$$\alpha : G \times \mathbb{C}[X] \longrightarrow \mathbb{C}[X], \quad (g, f) \longmapsto \begin{matrix} (g \cdot f) : X \to \mathbb{C} \\ \quad\quad x \mapsto f(g^{-1} \cdot x) \end{matrix}$$

From now X will be an affine variety and G an affine algebraic group.

Lemma 7. *The vector space $\mathbb{C}[X]$ is a union of finite dimensional $G-$stable subspaces on which G acts algebraically.*

Proof. We consider the pullback

$$\alpha^{\#} : \mathbb{C}[X] \longrightarrow \mathbb{C}[G \times X] \simeq \mathbb{C}[G] \otimes \mathbb{C}[X], \quad f \longmapsto (f \circ \alpha),$$

where $f \circ \alpha : G \times X \to \mathbb{C}$ and $f \circ \alpha(g, x) = f(g \cdot x)$. We write $\alpha^{\#}(f) = f \circ \alpha = \sum_{i=1}^{n} \varphi_i \otimes \psi_i$, hence

$$\alpha^{\#}(f)(g, x) = f(g \cdot x) = \sum_{i=1}^{n} \varphi_i(g)\psi_i(x), \quad \psi_i \in \mathbb{C}[X]$$

But we know that $g \cdot f = f(g^{-1} \cdot \_) = \sum_{i=1}^{n} \varphi_i(g^{-1})\psi_i(\_)$, so the translates $g \cdot f$ span a finite-dimensional subspace $V \subseteq \mathbb{C}[X]$ containing f, on which G acts, that is clearly $G-$stable. $\square$

The pullback $\alpha^{\#}$ is also called *coaction homomorphism*. This Lemma leads to the following definition.

Definition 8. A *(rational) G-module* is a complex vector space V equipped with a linear action of G such that every $v \in V$ is contained in a finite-dimensional $G-$stable subspace on which G acts algebraically.

Example 9. Coordinate rings of $G-$varieties are rational $G-$modules by Lemma 7. $\diamond$

Remark 10. Finite dimensional $G-$modules are in $1:1$ correspondence with finite dimensional algebraic representations of G. Indeed for a finite dimensional $G-$module V we can construct the representation $\rho : G \to \mathrm{GL}(V)$ as follows:

$$\rho(g) : V \longrightarrow V, \quad x \longmapsto g \cdot x.$$

Conversely, a finite-dimensional representaton $\rho : G \to \mathrm{GL}(V)$ yields the action

$$\cdot : G \times V \longrightarrow V, \quad (g, x) \longmapsto \rho(g)(x).$$

Example 11. One can prove that $\mathbb{C}^*-$modules corresponds to $\mathbb{Z}-$graded vector spaces. ◇

Definition 12. Give two $G-$varieties X, Y, we say that a morphism $f : X \to Y$ is *equivariant*, or that it is a *G-morphism*, if $f(g \cdot x) = g \cdot f(x)$ for every $g \in G$ and $x \in X$.

Proposition 13. *Let G be an affine algebraic group and X an affine $G-$variety. Then X is equivariantly isomorphic to a closed $G-$subvariety in a finite-dimensional $G-$module.*

Proof. We choose generators $\mathbb{C}[X] = \mathbb{C}[f_1, \dots, f_n]$. By Lemma 7 $\{g \cdot f_i \in \mathbb{C}[X] | \, g \in G\} \subseteq V_i \subset \mathbb{C}[X]$, where V is a finite-dimensional $G-$module. It follows that $V \colon = V_1 + \cdots + V_n$ generates the algebra $\mathbb{C}[X]$ and the map

$$\iota : X \longrightarrow V^*, \quad x \longmapsto (v \mapsto v(x))$$

is a closed equivariant immersion. □

Definition 14. Given a $G-$variety X and $x \in X$ we define the *orbit* of x and the *stabilizer* as

$$G \cdot x \colon = \{g \cdot x \mid g \in G\}, \quad G_x \colon = \{g \in G \mid g \cdot x = x\}.$$

The stabilizer is a closed subgroup of G, also called *isotropy group of x*.

Proposition 15. *We have:*

1. *The orbit $G \cdot x$ is a locally closed smooth subvariety of X.*
2. *Every component of $G \cdot x$ has dimension $\dim(G) - \dim(G_x)$.*
3. $\overline{G \cdot x} = G \cdot x \cup$ *smaller dimension orbits.*

Example 16.

1. $\mathbb{C}^*$ acts on $\mathbb{C}^n$ by $t \cdot (x_1, \dots, x_n) := (tx_1, \dots, tx_n)$. Then $\{0\}$ is the unique closed orbit and the other orbits are the lines through 0.
2. $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The closed orbits are the origin and the hyperbolae $\{xy = c\}$.
3. SL_2 acts on $\mathbb{C}^2$ by multiplication. The orbits are the origin and its complement, that is not an affine variety. The stabilizer of $(1, 0)$ is $U_2 \simeq \mathbb{C}$.

◇

Corollary 17. *Any affine algebraic group is linear.*

Proof. Let G be an affine algebraic group acting on itself by multiplication. The algebra $\mathbb{C}[G]$ is generated by a finite-dimensional $G-$module V, so $G \to \mathrm{GL}(V)$ is injective and the image is closed by Proposition 3, so G can be seen as a closed subgroup of $\mathrm{GL}(V)$, hence it is linear. □

Now we state the result that is the starting point for construction of quotients of linear algebraic groups by closed subgroups.

Proposition 18. *Every closed subgroup $H \subseteq G$ of a linear algebraic group is the stabilizer of a line ℓ in a finite-dimensional $G-$module V.*

This result can be rephrased as *any closed subgroup of G is the stabilizer of a point in the projectivization of a finite dim. $G-$module.*

Theorem 19. *Let G be a linear algebraic group and H a closed subgroup. Then G/H has a unique structure of $G-$variety that satisfies the following properties:*

(i) The quotient map $\pi : G \to G/H$, $g \mapsto gH$ is a morphism.

(ii) $U \subseteq G/H$ is open if and only if $\pi^{-1}(U)$ is open in G.

(iii) For any open $U \subseteq G/H$ the pullback $\pi^{\#} : \mathbb{C}[U] \simeq \mathbb{C}[\pi^{-1}(U)]^H$ is an isomorphism.

Moreover, G/H is smooth and quasi-projective.

Proof. (sketch) By Proposition 18 there exists a $G-$module V and $x \in \mathbb{P}(V)$ such that $H = G_x$. Let $X := G \cdot x$ and $p : G \to X$, $g \mapsto g \cdot x$ the orbit map. Then p is a surjective $G-$morphism whose fibers are the cosets gH. By generic smoothness and equivariance, π is a smooth open morphism satisfying (i) and (ii). □

Definition 20. A variety X is *homogeneous* if it is equipped with a transitive action of an algebraic group G. A *homogeneous space* is a pair (X, x), where X is homogeneous and x is the *base point*.

By the previous Theorem, it follows that homogeneous spaces (X, x) with an action of a linear algebraic group G are exactly the quotientspaces G/H, where $H = G_x$.

2 Reductive groups

Now two questions naturally arise, one more geometric, one more algebraic.

Question 21. Given an action of an algebraic group G on a $G-$variety, what is the correct definition of a *geometric quotient*? Does it always exists?

Question 22 (Hilbert's 14 problem)**.** An action of an algebraic group G on X induces an action on the polynomial algebra $\mathbb{C}[X]$ and give rise to the invariant functions $\mathbb{C}[X]^G$. More in general, if a group G acts over a finitely generated $k-$algebra A, is the invariant ring A^G finitely generated?

The answer to both questions is negative. In particular, Nagata gave a counter-example of an action of an affine algebraic group for which the ring of invariant is not finitely generated. However, the restriction on the hypothesis to have an affirmative answer to both questions is the same, given by reductive groups!

We will see more precisely how to construct geometric quotients, now we will give the tools that ensure their existence. In particular, the existence of a geometric quotient depends also by the properties of the acting group G.

Altough we assumed to work over $\mathbb{C}$, we give the definition of reductive group over any field k, to underline the importance of this assumption in the examples.

Definition 23. A linear algebraic group G is *reductive* over k if it does not contain any closed normal unipotent subgroup, i.e. a subgroup isomorphic to U_n.

We want to characterize reductive groups through their linear representations.

Definition 24. Let G be an algebraic group and V a $G-$module ($\rho : G \to \mathrm{GL}(V)$ a representation on a vector space V) . We say that V (ρ) is *simple* (or *irreducible*) if it has no proper $G-$submodules (V has no proper subspaces stable under $\rho(G)$). V is *semi-simple* (or *completely irreducible*) if it satisfies one of the equivalent condition:

(i) V is sum of simple submodules.

(ii) $V \simeq \bigoplus V_i$, where V_i are simple $G-$modules (V splits into direct sum of simple subspaces that are invariant under $\rho(G)$).

(iii) Any submodule $W \subseteq V$ admits a $G-$stable complement s.t. $V = W \oplus W'$

Example 25. Let G be a unipotent group, then every simple $G-$module is trivial, i.e. isomorphic to $\mathbb{C}$. ◇

The previous definition can be restated in terms of representations of G.

Theorem 26. *The following are equivalent for a linear algebraic group G over $\mathbb{C}$:*

(i) G is reductive.

(ii) G contains no closed normal subgroup isomorphic to $(\mathbb{C}^n, +)$.

(iii) G has a compact subgroup K that is Zariski dense.

(iv) Every finite-dimensional $G-$module is semi-simple. (definition of linearly reductive *group over any field k , equivalent to reductive in characteristic 0)*

(v) Every $G-$module is semi-simple.

Part (*iv*) can be restated as follows: every linear representation $\rho : G \to \mathrm{GL}(V)$ is completely reducible, that is, it decomposes as a direct sum of irreducible representations. Equivalently, the $\mathbb{C}-$vector space V splits into the direct sum of irreducible $G-$modules.

For completeness we report the following.

Definition 27. An affine algebraic group G is *geometrically reductive* is for every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every $v \in V^G \setminus 0$, there exists a $G-$invariant non-constant homogeneous polynomial $f \in k[V]$ such that $f(v) \neq 0$.

Theorem 28 (Weyl, Nagata, Mumford, Haboush)**.** *For smooth affine algebraic group:*

$$\textit{linearly reductive} \Rightarrow \textit{geometrically reductive} \Leftrightarrow \textit{reductive}$$

These notions coincide in characteristic zero.

Example 29.

1. GL_n, SL_n and Sp_{2n} are reductive if $\mathrm{char}(k) = 0$. They are not linearly reductive in positive characteristic.
2. The algebraic torus $(\mathbb{C}^*)^n$ is reductive.
3. The Borel subgroup of GL_n is not reductive since it corresponds to upper triangular matrices, and hence contains U_n.
4. Any finite group whose order is not divisible by the characteristic of k is linearly reductive. In particular, if $\mathrm{char}(k) = 0$, every finite group is reductive.

◇

Definition 30. We say that an algebraic group G acts *rationally* on a finitely generated $k-$algebra Aif every element of A is contained in a finite dimensional $G-$invariant linear subspace of A

Theorem 31 (Nagata)**.** *Let G be a reductive group acting rationally on a finitely generated $k-$algebra A. Then the invariant algebra A^G is finitely generated.*

Corollary 32. *If $A = \mathbb{C}[X]$, where X is a $G-$variety and G an affine algebraic group, then $\mathbb{C}[X]^G$ is finitely generated.*

To prove Nagata's Theorem we need the following tools/results.

Definition 33. Let G be a group acting on a $k-$algebra A. A *Reynolds operator* is a map of A^G-modules $\rho : A \to A^G$ such that $\rho_{|A^G} = \mathrm{id}_{A^G}$.

Lemma 34. *A reductive group G acting rationally on a finitely generated algebra A admits a Reynolds operator.*

Corollary 35. *If a reductive group G acts rationally on algebras A and B that admit a Reynolds operator, then any $G-$equivariant morphism $f : A \to B$ commutes with the Reynolds operators*

$$\rho_B \circ f = f \circ \rho_A.$$

Lemma 36. *In the same hypothesis, for every ideal $I \subseteq A^G$, we have $IA \cap A^G = I$. In particular, if A is Noetherian then so is A^G.*

Proof. (Nagata's Theorem)

First we reduce to the case where A is a polynomial algebra with a linear $G-$action.

We consider $A = k[f_1, \ldots, f_c]$. Since the action is rational, for every f_i we can find a finite dimensional $G-$module V_i containing f_i. In particular we find a finite dimensional $G-$module V containing the algebra generators $f_1, \ldots, f_c$. This gives us a $G-$equivariant surjection

$$\mathrm{Sym}^*(V) \twoheadrightarrow A \qquad i.e.(k[x_1, \ldots, x_c] \twoheadrightarrow A)$$

between algebras that admit a Reynolds operator by Lemma 34. By Corollary 35 we have a surjection between the invariant rings:

$$\mathrm{Sym}^*(V)^G \twoheadrightarrow A^G \qquad i.e.(k[x_1, \ldots, x_c]^G \twoheadrightarrow A)$$

So if we prove that $k[x_1, \ldots, x_c]^G$ is finitely generated we can conclude.

Let us assume that $A = k[x_1, \ldots, x_c]$. In particular A is standard graded by: $A = \bigoplus_{n \geq 0} A_n$. The invariant ring A^G inherits the grading

$$A^G = \bigoplus_{n \geq 0} (A^G)_n = \bigoplus_{n \geq 0} (A_n)^G.$$

By the previous Lemma, A^G is Noetherian, so the maximal homogeneous ideal $A^G_+ = \bigoplus_{n > 0} (A^G)_n$ is finitely generated. Let $a_1, \ldots, a_m$ be generators of A^G_+, it follows, from a general result in commutative algebra, that $A^G = A_0[a_1, \ldots, a_m] = k[a_1, \ldots, a_m]$, so it is a finitely generated $k-$algebra.

□

References

[1] Michel Brion. *Introduction to actions of algebraic groups.*

[2] Victoria Hoskins. *Moduli problems and geometric invariant theory.*