Affine GIT

In this talk, we will define several notions of quotients for affine varieties and study how they are related to each other. Here, the group G will always be an affine algebraic group.

Definition 1. Let G be an affine algebraic group acting on a variety X. A categorical quotient is a variety Y and a G-invariant morphism $\pi : X \to Y$ satisfying the following universal property. For any variety Z, and for any G-invariant morphism $f : X \to Z$, f factors uniquely through π: $f : X \xrightarrow{\pi} Y \to Z$.
In this case, we denote Y by $X//G$.

This definition is large, but it is difficult to compute a quotient. The second notion of quotient will be equivalent if everything is affine and allows us to compute quotients.

Definition 2. Let G be a reductive group acting on an affine variety X. The affine GIT quotient, denoted $X//G$ is the image of the morphism $\pi : X \to \mathbb{C}^n, x \mapsto (f_1(x), ..., f_n(x))$, for $f_1, ..., f_n$ generators of $\mathbb{C}[X]^G$.

The surjective map $\pi : X \to X//G$ corresponds to the inclusion $\mathbb{C}[X]^G \subseteq \mathbb{C}[X]$. Equivalently, $X//G = Spec(\mathbb{C}[X]^G)$.

Lemma 3. *The image $X//G$ is closed and independent of the generators $f_1, ..., f_n \in \mathbb{C}[X]^G$.*

Definition 4. The affine GIT quotient is a categorical quotient.
Moreover, if G is reductive, and the categorical quotient is affine, then it is an affine GIT quotient.

Proposition 5. *Let G be a reductive group and X be an affine variety such that $X//G$ is affine. Then:*

1. *Let $Z \subseteq X$ be a G-invariant closed subspace. Then, $f : Z//G \to X//G$ is a closed immersion (i.e. $f^{\#} : \mathbb{C}[X]^G \to \mathbb{C}[Z]^G$ is surjective).*
2. *Let $Z, Z' \subseteq X$ be G-invariant closed subspaces. Then, $\pi(Z \cap Z') = \pi(Z) \cap \pi(Z')$.*
3. *Each fiber of π contains a unique G-orbit.*
4. *If X is irreducible/normal, then so is $X//G$.*

Example 6.

- Let $\mathbb{C}^*$ acts on $(\mathbb{C})^n$ by $(t, (x_1, ..., x_n)) \mapsto (tx_1, ..., tx_n)$. Then $\mathbb{C}^*$ acts on the coordinate ring $\mathbb{C}[x_1, ..., x_n]$ by $(t, f(x_1, ..., x_n)) \mapsto f(t^{-1}x_1, ..., t^{-1}x_n)$. The invariant elements are the constant elements: $\mathbb{C}[x_1, ..., x_n]^G = \mathbb{C}$. The categorical quotient is just a point $(\mathbb{C})^n//\mathbb{C}^* = \{0\}$.
- Let $\mathbb{C}^*$ acts on $(\mathbb{C})^2$ by $(t, (x_1, x_2)) \mapsto (tx_1, t^{-1}x_2)$. Similarly, $\mathbb{C}^*$ acts on the coordinate ring $\mathbb{C}[x_1, x_2]$ by $(t, f(x_1, x_2)) \mapsto f(t^{-1}x_1, tx_2)$. In this case, we have non-constant invariant elements: $\mathbb{C}[x_1, x_2]^G = \mathbb{C}[x_1x_2] = \mathbb{C}[z]$. So the categorical quotient is $\mathbb{C}^2//\mathbb{C}^* = \mathbb{A}^1$.
  In this case, the three orbits $\{x = 0\}, \{y = 0\}$ and $\{(0, 0)\}$ are sent to the point $0 \in \mathbb{A}^1$. However, it doesn't contradict Proposition 5 (3) because only $\{(0, 0)\}$ is closed.

We see from these examples that the categorical/affine GIT quotient doesn't allow us to compute many quotients. In the first example, we would like to consider the same action on the space $(\mathbb{C})^n \setminus \{0\}$ to get $\mathbb{P}^{n-1}$. However, since $\mathbb{P}^{n-1}$ is not affine, this will not be covered in this talk but in the one about projective GIT.

Definition 7. Let G be an affine algebraic group acting on a variety X. A good quotient is a variety Y, together with a G-invariant morphism $\pi : X \to Y$ such that:

1. π is surjective, and for $W, W' \subseteq X$ disjoint, G-inavariant closed subspaces, $\pi(W) \cap \pi(W') = \emptyset$.

2. A subspace $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open.

3. For any open subspace $U \subseteq Y$, π yields an isomorphism $\mathbb{C}[\pi^{-1}(U)]^G \simeq \mathbb{C}[U]$.

Proposition 8. *Let G be an affine algebraic group acting on a variety X, and Y be a good quotient. Then, Y is a categorical quotient.*
Conversely, if G is reductive, X is affine, and Y is an affine categorical quotient, then Y is a good quotient.

Proposition 9. *Let G be an affine algebraic group. A geometric quotient of X by G consists of a variety Y together with a morphism $f : X \to Y$ such that:*

1. *π is surjective, and its fiber are exactly the G-orbits in X.*

2. *A subspace $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open.*

3. *For any open subspace, π yields an isomorphism $\mathbb{C}[\pi^{-1}(U)]^G \simeq \mathbb{C}[U]$.*

If it exists, we denote Y by X/G.

The geometric quotients are actually the quotients we would like to consider. The problem is that they don't always exist.

Proposition 10. *If Y is a geometric quotient, then it is also a good and a categorical quotient.*

Proposition 11. *If $\pi : X \to Y$ is a good/geometric quotient, then for every $U \subseteq Y$ open, $\varphi_{|\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is also a good/geometric quotient.*
Moreover, if $\pi : X \to Y$ is G-invariant, and we have a cover of Y by open subspaces U_i such that $\varphi_{|\varphi^{-1}(U_i)} : \varphi^{-1}(U_i) \to U_i$.

Definition 12. Let G be a reductive group acting on an affine variety X such that $X//G$ is affine. A point $x \in X$ is stable if $G \cdot x$ is closed and G_x is finite (i.e. $\dim(G_x) = 0$).
We denote the set of stable points of X by X^s.

Proposition 13. *The projection $\pi(X^s)$ is open in $X//G$ and $X^s = \pi^{-1}(\pi(X^s))$. In particular X^s is open and G-invariant. The restriction map $\pi^s : X^s \to \pi(X^s)$ is a geometric quotient.*

Example 14.

- Let $\mathbb{C}^*$ act on $\mathbb{C}^n$ as in Example 6. This action does not admit a geometric quotient as the only closed orbit is $\{0\}$, and $\{0\}$ belongs to all the orbit closures.
  Moreover, there are no stable points as the stabilizer of $\{0\}$ is infinite, and none of the other orbits are closed.
  In this case, we can compute the quotient of $\mathbb{C}^n \setminus \{0\}$, under the action of $\mathbb{C}^*$, and we will get $\mathbb{P}^{n-1}$, as every line is precisely identified to a point.

- Let $\mathbb{C}^*$ act on $\mathbb{C}^n$. The affine GIT quotient is not a geometric quotient as the orbit $\{x = 0\}$ and $\{y = 0\}$ are not closed.
  The stable points are $X^s = \mathbb{C}^2 \setminus (\{x = 0\} \cup \{y = 0\})$. The geometric quotient $X^s/\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

is given by the map $X^s \to \mathbb{C} \setminus \{0\}, (x, y) \mapsto xy$.
Let $G = GL_2$ be the group of 2×2 invertible matrices acting on the variety of 2×2 matrices $Mat_{2\times 2}$ by conjugation, $(G, M) \mapsto GMG^{-1}$. The orbits are given by the Jordan canonical form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}.$$

The orbit corresponding to the second type of matrices is closed; the orbit of the first kind is just a point, but the closure of the orbit of the third contains the first type. Moreover, there are no stable points as $t \cdot id$ is in the stabilizer of all matrices. We don't have a GIT quotient. We can prove that $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]^G = \mathbb{C}[tr, det] \subseteq \mathbb{C}[y_1, y_2]$. The affine GIT quotient is given by the map $Mat_{2\times 2} \to \mathbb{A}^2, M \to (tr(M), det(M))$.