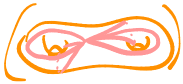
The character variety

(or the Betti moduli space)

focus for later: $\pi_1\left(\right.$  $\left.\right) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle$

or of some other compact mfld (Kähler would be perfect)

$$\mathcal{M}_B := \underbrace{\mathrm{Hom}(\Gamma, G)}_{\text{1. affine alg. variety}} /\!\!/_{GIT}\ G$$

Γ: finitely generated gp

G: complex alg. reductive

G acting by conjugation

2. How to take GIT.

1. The variety $\mathrm{Hom}(\Gamma, G)$

Let G be a reductive complex algebraic group (e.g. $GL(n, \mathbb{C})$)

recall: If it contains no normal proper subgroups $\cong U_n$

$\iff Ad(G) < GL(\mathfrak{g})$ is completely reducible

$\iff Ad(G) < GL(\mathfrak{g})$ splits into irreducible reps as a direct sum

For $GL(n, \mathbb{C})$, $\mathfrak{g} = Mat_{n \times n}(\mathbb{C})$, and $Ad(g)(X) = gXg^{-1}$.

The representation variety is $\mathrm{Hom}(\Gamma, G)$ w/ subspace top. of G^Γ (w/ compact-open top.)

↳ need to interpret this as an algebraic variety

Say $\Gamma = \langle \gamma_1, \dots, \gamma_n \mid \{r_i\} \rangle$, then set

$$X(\Gamma, G) := \{ \rho(\gamma_1), \dots, \rho(\gamma_n)) : \rho \in \mathrm{Hom}(\Gamma, G) \}$$
$$\subseteq G^n.$$

and of course $X(\Gamma, G) \cong \mathrm{Hom}(\Gamma, G)$

Lemma $X(\Gamma, G)$ is an algebraic subset of G^n, i.e. $\mathrm{Hom}(\Gamma, G)$ has the structure of an algebraic variety & the structure doesn't depend on the generators.

Pf idea: The relations give rise to algebraic regular maps (the word map)

$$r_i : G^n \to G$$

$$X(\Gamma, G) = \{ (g_1, \dots, g_n) \in G^n : r_i(g_1, \dots, g_n) = 1 \}.$$

$\square$

$\mathrm{Hom}(\Gamma, G)$ carries an action of $\mathrm{Inn}(G) := G/Z(G)$ by conjugation

$$(g.\rho)(\gamma) = g \rho(\gamma) g^{-1}.$$

(we only care about ρ up to conj., because this is a change of basis).

And we'd like to build the quotient $\mathrm{Hom}(\Gamma, G)/G$.

Why the GIT quotient

To have a nice quotient, we'd want a free & properly discontinuous action.

Freeness: $\rho = id$ is a global fixed point. In fact

Prop (Goldman '84) The $\mathrm{Inn}(G)$-action on $\mathrm{Hom}(\Gamma, G)$ is locally free (stabilizer of pt's is discrete) iff

$$\dim Z(G) = \dim Z(\rho)$$

Moreover, if $\Gamma = \pi_1(S_g)$, this condition also implies ρ is smooth)

(Hausdorffness)

$$: \Gamma = \langle a, b \rangle, \quad \rho_1 : \Gamma \to SL(2,\mathbb{R}), \quad \rho_2 = id.$$
$$a \mapsto \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$
$$b \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^t \end{pmatrix} = \begin{pmatrix} e^t & e^t \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^t \end{pmatrix} = \begin{pmatrix} 1 & e^{2t} \\ 0 & 1 \end{pmatrix} \text{ as } t \to -\infty, \text{ get to id.}$$

$$\Longrightarrow \overline{G.\rho_1} \cap \underbrace{\overline{G.\rho_2}}_{=\{\rho_2\}} \neq \emptyset \Rightarrow \text{quotient would not be Hausdorff}$$

1. The GIT quotient

For this, we have to look at $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ & the invariant functions there. When G is an alg. lin. subgroup of $GL(n,\mathbb{C})$, there is a class of functions which is invariant:

Trace functions: Fix $\gamma \in \Gamma$,

$$\begin{aligned} tr_\gamma : \mathrm{Hom}(\Gamma, G) &\to \mathbb{C} \\ \rho &\mapsto tr(\rho(\gamma)) \end{aligned}$$

are invariant under the $\mathrm{Inn}(G)$-action.

Thm (Procesi '76) Let $G = GL(n,\mathbb{C}), SL(n,\mathbb{C}),$
$\underset{n\geq 2}{O(n,\mathbb{C})^\circ}, \underset{n\geq 3}{SO(n,\mathbb{C})}, Sp(2n,\mathbb{C})$.

Then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is generated by trace functions (generated as an algebra by $\{tr_\gamma \mid \gamma \in \Gamma\}$).

From Anaëlle's talk, we needed to have

Thm (Nagata) $\underset{G \text{ reductive}}{\wedge}$ $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated.

So we can take the affine GIT quotient

$$\mathrm{Hom}(\Gamma, G) /\!\!/_{GIT} G := \mathrm{Spec}\left(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G\right).$$

Can do it like this (Anaëlle): say $f_1, \dots, f_e$ generate $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$, then the image of the map

$$\pi : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathbb{C}^n$$
$$\rho \longmapsto (f_1(\rho), \dots, f_e(\rho))$$

$\underline{\underline{\text{IS}}}$ the GIT quotient, also write

$$\pi : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) /\!\!/_{GIT} G.$$

Remarks: 1) $\mathcal{O}_1 := G.\rho_1$, $\mathcal{O}_2 := G.\rho_2$ get identified if

$$\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2} \neq \emptyset$$

(any $f \in \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is constant on $\overline{\mathcal{O}_1}, \overline{\mathcal{O}_2} \Rightarrow$ any f takes the same value on $\mathcal{O}_1$ & $\mathcal{O}_2$)

2) Since $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is generated by trace fct's.

$$\mathrm{Hom}(\Gamma, G) /\!\!/_{GIT}\, G \cong \mathrm{Hom}(\Gamma, G) \Big/ \begin{array}{l} \rho_1 \sim \rho_2 \\ \Leftrightarrow \mathrm{tr}_\gamma(\rho_1) = \mathrm{tr}_\gamma(\rho_2) \\ \quad \forall \gamma \in \Gamma \end{array}$$

(This is a very useful way to think about character varieties!)

To understand the quotient a little better, recall that each fibre of π contains a unique closed orbit

Def.: $\rho \in \mathrm{Hom}(\Gamma, G)$ is **polystable** if its orbit $\mathcal{O}_\rho$ is closed.

Thm (Sikora) For any reductive alg. group G,

ρ is polystable $\Longleftrightarrow$ ρ is completely reducible (& each fibre contains a reductive representation.)

(also called reductive)

Equivalent definitions of c.r. 1. ρ decomposes as a direct sum of irreducible representations

1. For every parabolic subgroup $P < G$ w/ $\rho(\Gamma) < P$,

G/P is a projective variety

P contains a Borel subgroup

max. $\mathcal{Z}$. closed solvable connected

$G = GL(n, \mathbb{C})$, P is (up to conj.) block upper triangular

i.e. a stab. of a (partial) flag

$$P = \left\{ \begin{pmatrix} \boxed{*} & * & * \\ 0 & \boxed{*} & * \\ 0 & 0 & \boxed{*} \end{pmatrix} \right\}$$

there is a Levi subgroup $L \subset G$ containing $\rho(T)$

↳ a centralizer of a subtorus of G

↳ abelian subgroup of G.

↳ $G = GL(n,\mathbb{C})$, the stabilizer of direct sum decompositions $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$ (block diagonal)

$$L = \left\{ \begin{pmatrix} \boxed{*} & 0 & 0 \\ 0 & \boxed{*} & 0 \\ 0 & 0 & \boxed{*} \end{pmatrix} \right\}$$

→ in particular, completely reducible representations cannot be upper triangular, so we get rid of the bad example above!

Thus, the theorem implies that

alg. variety! & $\mathrm{Inn}(G)$-inv.

usual topological quotient

$$\mathrm{Hom}(T,G)/\!\!/_{GIT}\, G \cong \mathrm{Hom}^{red}(T,G)/G.$$

Hausdorff!

There is another type of representations still

in this case, if $\dim Z(G) = \dim Z(\rho)$

Def.: ρ is stable if it is polystable & a smooth point of $\mathrm{Hom}(T,G)$

(or if $\exists$ a Zariski open nbhd of ρ preserved by G on which the G action is closed) OR if polystable & finite stabilizer

Thm (Sikora) ρ is stable $\Leftrightarrow$ ρ is irreducible & $C(G)$ is finite.

Moreover $\mathrm{Hom}^{irr}(\Gamma, G)$ is Zariski open & an alg. variety & $\mathrm{Inn}(G)$-inv.

$\Rightarrow$ dense in analytic topology.

And so $$\mathrm{Hom}^{s}(\Gamma, G)/\mathrm{Inn}(G) = \underbrace{\mathrm{Hom}^{irr}(\Gamma, G)}_{\text{smooth manifold}}/\mathrm{Inn}(G)$$

$\nearrow$ a topological quotient

space of reductive representations is algebraic

Cor: When G is a complex reductive algebraic group, $\mathrm{Hom}(\Gamma, G) /\!\!/_{GIT}\, G$ is an algebraic variety.

(Thm (Richardson-Slodowy '90) When G is a real algebraic group, $\mathrm{Hom}^{red}(\Gamma, G)/\mathrm{Inn}(G)$ is a real semialgebraic set (in general)

$\hookrightarrow$ polynomial inequalities