de Rham Moduli Space: Flat Connections

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Σ: closed surface G: complex reductive Lie group, usually $GL(n,\mathbb{C})$ $\rho : \pi_1(\Sigma) \to G$

Manifolds and maps are considered to be in the smooth category (not holomorphic).

De Rham moduli space is the space of flat connections over $\Sigma \times G$ up to gauge transformation.

Given a reprensentation $\rho : \pi_1(\Sigma) \to G$, we can construct the following bundle

$$\tilde{\Sigma} \times_\rho G := \tilde{\Sigma} \times G/\sim \quad (x, g) \sim (x.\gamma^{-1}, \rho(\gamma).g) = (\gamma.x, \rho(\gamma).g)$$

It's a $G-$principal bundle over Σ. When $G = GL(n,\mathbb{C})$, principal bundles are in some sense equivalent to associated complex rank n vector bundles. Such complex vector bundles are classified by degree and rank, so it does not matter to consider $\Sigma \times G$ or $\tilde{\Sigma} \times_\rho G$. The set of connections is actually the set of different holomorphic structure on this vector bundle.

The goal of this talk is to define a symplectic structure on de Rham moduli space. Here is the motivation: the moduli space is a natural example of hyperkähler manifold, which is a Riemannian manifold equipped with three integrable almost complex structure I, J, K such that they are kähler with respect to the Riemannian structure (which also gives you a symplectic structure) and

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}$$

Back to our case, there are three different ways of realizing the moduli spaces: Betti, de Rham, Dolbeault. Each one comes with its own symplectic structure, and this talk will focus on the symplectic structure on de Rham moduli spaces.

1 Principal Bundle

Definition 1.1. Given a fibre bundle $\pi : P \to M$ with fibre diffeomorphic to a Lie group G. We say it is a principal G-bundle if it is equipped with a right $G-$action such that G acts freely and transitively on each fibre.

Example 1.2. Here are some basic examples of principal bundles.

- $\mathbb{R}^2 \to \mathbb{R}$
- $T^1\Sigma \to \Sigma$, here $T^1\Sigma$ is the unit tangent bundle

- $\tilde{\Sigma} \to \Sigma$
- frame bundle

Definition 1.3. The gauge group $\mathcal{G}$ is the $G-$equivariant automorphism group of P. More explicitly,

- $\pi \circ \phi = \pi$
- $\phi(pg) = \phi(p)g$

Remark. Since ϕ preserves the fibre, we could write it as $\phi(p) = p \cdot f(p)$ for a map $f : P \to G$. Using the $G-$equivariance of ϕ, we can deduce that

$$f(pg) = g^{-1} f(p) g$$

Thus $\mathcal{G} = \mathrm{Aut}^G(P) = \mathrm{Hom}^G(P, G)$.

Definition 1.4. A connection on a principal bundle P is an element $A \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$ such that

- $R_g^* A = \mathrm{Ad}_{g^{-1}} A$
- take $p \in P, \xi \in \mathfrak{g}$, we can define

  $$\sigma_p(\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \cdot \exp t\xi \in T_p P$$

  these are called vertical vectors. The collection of all vertical vectors form a subbundle of TP called vertical distribution, denoted by VP.

  A is required to preserve the vertical vector field: $A_p \circ \sigma_p = \mathrm{Id}$

Given a connection, $A^{-1}(0)$ is a subbundle of TP, which is called the horizontal distribution, denoted by HP. Roughly speaking, VP assigns each point p in the base manifold a vector space $\mathfrak{g}$, and as for HP it is $T_p M$. Note that the definition of VP does not rely on the connection, while HP does. A connection is equivalent to the choice of a horizontal distribution.

Fix a principal bundle, we denote the set of all connections by $\mathcal{A}$. It is an affine space with the underlying vector space to be $(\Omega^1_h(P, \mathfrak{g}))^G$, the set of G equivariant $\mathfrak{g}$ valued 1 forms which vanish on vertical vectors. Here "h" means "horizontal".

Definition 1.5. The curvature form of a connection A is

$$F_A := \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g})$$

A connection with zero curvature form is called a flat connection.

The gauge group naturally acts on the set of connections by $\phi.A = \phi^* A$ and also the curvature $F_{\phi.A} = \phi^*(F_A)$. From this formula, we can get the following result

Proposition 1.6. *The set of flat connections is preserved by the gauge group action.*

2 Symplectic Geometry

Definition 2.1. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a nondegenerate closed 2 form.

We know from the definition that at each point, ω restrics to a nondegenerate skew-symmetric bilinear form. In particular, the manifold must be even dimensional.

Example 2.2. Here are some examples of symplectic manifolds.

- $\mathbb{R}^{2n}$ $\quad \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$
- $\mathbb{C}^n$ $\quad \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n)$
- $\mathbb{C}P^n$ $\quad$ Fubini-Study form
- The set of flat connections for a fixed principal bundle over a surface $\mathcal{A}$

  Take $A \in \mathcal{A}$. Since $\mathcal{A}$ is an affine space, we can identify the tangent space $T_A\mathcal{A}$ with the underlying vector space $(\Omega^1_h(P, \mathfrak{g}))^G$. Take two vectors $B, C \in (\Omega^1_h(P, \mathfrak{g}))^G$, define a 2 form in the following way:

$$\omega_A(B, C) = \int_\Sigma \mathrm{tr}\ B \wedge C$$

  Here $B \wedge C$ is a horizontal 2 form valued in $\mathfrak{g} \otimes \mathfrak{g}$. When $G = GL(n, \mathbb{C})$, tr is just the trace form of the product of two matrices. In genearl, it is the so-called Killing form, which is an Ad invariant biliear form on Lie algebras. Note that it is a horizontal form which vanishes if any entry is a vertical vector, so it reduces to a 2 form on the base manifold Σ, and the integration makes sense.

Definition 2.3. Given a function $f \in C^\infty(M)$, the symplectic gradient is defined by

$$\omega(\mathrm{sgrad}(f), X) = -\mathrm{d}f(X)$$

It also gives $C^\infty(M)$ a Lie algebra structure called Poisson bracket

$$\{f, g\} = \omega(\mathrm{sgrad}(f), \mathrm{sgrad}(g))$$

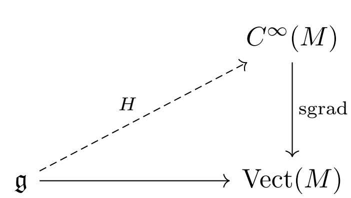
3 Symplectic Reduction

Definition 3.1. An action of G on a symplectic manifold (M, ω) is called symplectic if $g^*\omega = \omega$

Its infinitesimal $\mathfrak{g} \to \mathrm{Vect}(M)$ is a Lie algebra homomorphism, given by

$$X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}t} \exp t\xi.p$$

Definition 3.2. A symplectic action is called Hamiltonian if $\mathfrak{g} \to \mathrm{Vect}(M)$ can be lifted to a Lie algebra homomorphism $H : \mathfrak{g} \to C^\infty(M)$



Definition 3.3. A moment map $\mu : M \to \mathfrak{g}^*$ is defined by

$$\mu(p)(\xi) = H_\xi(p) \quad p \in M, \xi \in \mathfrak{g}$$

The moment map is G equivariant:

$$\mu(g.p) = \mathrm{Ad}_g^*(\mu(p))$$

where Ad^* is the coadjoint action on $\mathfrak{g}^*$.

Definition 3.4. Given a Hamiltonian action and the moment map. Given a coadjoint orbit $\mathcal{O} \in \mathfrak{g}^*$, if every point in $\mathcal{O}$ is a regular value of μ, then the Hamiltonian reduction is defined to be

$$M//_{\mathcal{O}}G := \mu^{-1}(\{\mathcal{O}\})/G$$

Theorem 3.5. *If G acts on $\mu^{-1}(\mathcal{O})$ is free and proper, then $M//_{\mathcal{O}}G$ is a symplectic manifold with symplectic structure inherited from the original manifold M.*

Theorem 3.6. *Consider the gauge group action on the space of connections: $M = \mathcal{A}, G = \mathcal{G}$. We have the following results:*

- *the action is Hamiltonian*
- *the moment map μ is exactly the curvature form*
- *0 is a regular value of the moment map*
- *$\mathcal{G}$ acts on $\mu^{-1}(0)$, which is the set of flat connections, freely and properly*

Thus, we can conclude that the space of flat connections inherits the symplectic structure from $\mathcal{A}$.

Example 3.7. Here is an application of the symplectic reduction. Consider $\mathbb{S}^1 = U(1)$ acts on $\mathbb{C}^{n+1}$ by $\lambda.(z_i) = (\lambda z_i)$. Then the Lie algebra $\mathfrak{u}(1)$ can be identified with $i\mathbb{R}$.

Recall that the moment map is given by

$$\mu(p)(\xi) = H_\xi(p) \quad p \in M, \xi \in \mathfrak{g}$$

Then its differential map $\mathrm{d}\mu_p : T_pM \to T_{\mu(p)}\mathfrak{g}^* \cong \mathfrak{g}^*$ satisfies

$$\mathrm{d}\mu(-)(\xi) = \mathrm{d}H_\xi(-) = \omega(X_\xi, -)$$

Take an element $\xi = ir \in \mathfrak{u}(1)$, $r \in \mathbb{R}$, then $X_\xi = (irz_i)$. Here (z_i) actually means the radial vector field, and (iz_i) is the vector field tangent to circles centered at origin. Thus

$$X_\xi = r\left(z_i\frac{\partial}{\partial z_i} - \bar{z}_i\frac{\partial}{\partial \bar{z}_i}\right)$$

The symplectic form on $\mathbb{C}^n$ is $\omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n)$, and we could compute that

$$\omega(X_\xi, -) = -\frac{r}{2}(z_i\mathrm{d}\bar{z}_i + \bar{z}_i\mathrm{d}z_i) = -\frac{r}{2}\ \mathrm{d}(\|(z_i)\|^2)$$

Then we integarte it and get

$$\mu(z_i) = -\frac{1}{2}\|(z_i)\|^2 + \text{constant}$$

Let us just take the constant to be 0. Since the group is commutative, every number will be an orbit itself. Moreover, every number except 0 is a regular value. Let us take $-\frac{1}{2}$ and

$$\mathbb{C}^{n+1}//_{-\frac{1}{2}}\mathbb{S}^1 = \mu^{-1}(-\frac{1}{2})/\mathbb{S}^1 = \mathbb{S}^{2n+1}/\mathbb{S}^1 \cong \mathbb{C}P^n$$

So $\mathbb{C}P^n$ inherits a symplectic structure from $\mathbb{C}P^{n+1}$. This is exactly the Fubini-Study form.