Riemann-Hilbert correspondence

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§0. Review: talking from symplectic quotients

Atiyah-Bott, Donaldson, Hitchin, ...

X smooth proj. curve. (= cpt. R.S.)

$$\begin{array}{ccc} \text{unitary flat bundles} & \longleftrightarrow & \text{unitary representations of } \pi_1(X) \\ \wr & & \wr \\ \text{flat bundles} & \longleftrightarrow & \text{representations of } \pi_1(X) \end{array}$$

<u>idea</u>: Fix E $\mathbb{C}$-v.b. of rk n

Newlander-Nirenberg : holo. str on E $\Leftrightarrow$ $\bar\partial$-type operator $\bar\partial_E$

$C^{\bar\partial} := \{\text{holo. structures } \bar\partial_E\}$ ∞-dim. $\mathbb{C}$-v.s.

More precisely, affine space modelled $A^{0,1}(\mathrm{End}(E))$.

$G_{\mathbb{C}} := A^0(GL(E))$ cplx gauge group

$G_{\mathbb{C}} \curvearrowright C^{\bar\partial}$ via $g \cdot \bar\partial_E := g \circ \bar\partial_E \circ g^{-1}$

<u>want</u>:

$C^{\bar\partial}/G_{\mathbb{C}}$, but this is not a "good" space

<u>idea</u>:

make $C^{\bar\partial}$ into a symplectic mfld (M, ω) with $G \curvearrowright M$

$\uparrow$ real cpt. $(G)_{\mathbb{C}} = G_{\mathbb{C}}$

& moment map $\mu : M \to \mathfrak{g}^*$

& $G \curvearrowright \mu^{-1}(0)$ freely

$\rightsquigarrow$ get symplectic quotient $\mu^{-1}(0)/G$ "good" space

How to do?

(1) fix h hermitian metric on E.

$C^h := \{ \nabla$ connection on E : $h(\nabla u, v) + h(u, \nabla v) = dh(u,v) \}$

space of h-unitary connections

$$T_\nabla C^h \cong A^1(\mathfrak{u}(E)) = \{ f \in A^1(End(E)) : h(fu, v) + h(u, fv) = 0 \}$$

Prop. $C^{\bar\partial} \cong (C^h, *)$ as cplx spaces

- $\bar\partial_E \longmapsto \partial_h + \bar\partial_E$ "chern connection"
- $*: A^1(\mathfrak{u}(E)) \longrightarrow A^1(\mathfrak{u}(E))$

$\alpha \longmapsto *\alpha$

define symplectic form on C^h:

$$\omega : A^1(\mathfrak{u}(E)) \times A^1(\mathfrak{u}(E)) \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \longmapsto -\int_X Tr(\alpha \wedge \beta)$$

$\rightsquigarrow (C^h, \omega)$ symplectic mfd. $=: (M, \omega)$

$G := A^0(U(E)) := \{ g \in A^0(GL(E)) : h(gu, gv) = h(u,v) \}$

"unitary gauge gp"

$\mathfrak{g} = T_eG = A^0(\mathfrak{u}(E))$

$= \{ g \in A^0(End(E)) : h(gu, v) + h(u, gv) = 0 \}$

$G \curvearrowright M$ as $g \cdot \nabla := g \circ \nabla \circ g^{-1}$

$\rightsquigarrow$ moment map $\mu : M \longrightarrow \mathfrak{g}^* \cong A^2(\mathfrak{u}(E)) := \{ g \in A^2(End(E)) :$

$h(gu, v) + h(u, gv) = 0 \}$

$A^0(\mathfrak{u}(E)) \times A^2(\mathfrak{u}(E)) \longrightarrow \mathbb{R}$ non-degenerate

$(g, \Omega) \longmapsto \int_X Tr(g\Omega)$

$\nabla \longmapsto F_\nabla = (\nabla)^2$

$\rightsquigarrow \mu^{-1}(0)/G$ symplectic quotient $=: M_{uf}(X, n)$

$$\mathcal{M}_{\mathrm{uf}}(X,n) \longleftrightarrow \mathrm{Rep}(\pi_1(X), U(n)) = \mathrm{Hom}(\pi_1(X), U(n))/U(n)$$

unitary Riemann-Hilbert

(2) $\mathcal{C}^{\mathrm{Flat}} := \mathcal{C}^h \times A^1(\mathfrak{u}(E))$

$(D, \Phi) \in \mathcal{C}^h \times A^1(\mathfrak{u}(E))$

$T_{(D,\Phi)}\mathcal{C}^{\mathrm{Flat}} = A^1(\mathfrak{u}(E)) \times A^1(\mathfrak{u}(E))$

$I(A, \varphi) := (*A, -*\varphi)$

$J(A, \varphi) := (-\varphi, A)$

$K(A, \varphi) = IJ(A, \varphi) = (-*\varphi, -*A)$

$$g((A_1, \varphi_1), (A_2, \varphi_2)) := -\int_X \mathrm{Tr}(A_1 \wedge *A_2 + \varphi_1 \wedge *\varphi_2)$$

$\rightsquigarrow (\mathcal{C}^{\mathrm{Flat}}, I, J, K, g)$ hyperkähler mfld.

$\rightsquigarrow \omega_I, \omega_J, \omega_K$

$(\mathcal{C}^{\mathrm{Flat}}, J) =: (\mathcal{M}, \omega_J)$ space of "cplex connections" on E

$\{\nabla = D + i\Phi\}$

$G = A^0(U(E)) \curvearrowright \mathcal{M}$

$(g, \nabla) := g \circ \nabla \circ g^{-1}$

$\rightsquigarrow$ moment map

$\tilde{\mu}_J : \mathcal{M} \longrightarrow \mathfrak{g}^* \cong A^2(\mathfrak{u}(E))$

$\nabla = D + i\Phi \longmapsto -F_D + \Phi \wedge \Phi + iD\Phi$

$\text{"}(D + i\Phi)^2\text{"} = \nabla^2$

$$\rightarrow \mu_J^{-1}(0)/G =: \mathcal{M}_{dR}(X,n)$$

$\mathcal{M}_{dR}(X,n) \longleftrightarrow \mathrm{Rep}(\pi_1(X), GL_n(\mathbb{C})) = \mathrm{Hom}(\pi_1(X), GL_n\mathbb{C}) /\!/ GL_n\mathbb{C}$

$=: \mathcal{M}_B(X,n)$

Riemann-Hilbert correspondence

§1. Motivation – Riemann-Hilbert problem (Hilbert's 21st problem)

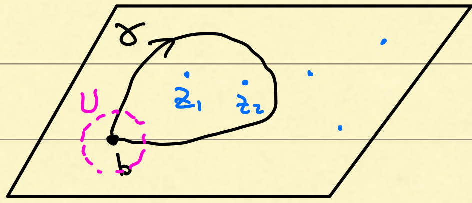
Consider the following system of linear ODEs on $\mathbb{C}\setminus\{z_1, \dots, z_k\}$

$$\frac{dY(z)}{dz} + A(z)Y(z) = 0$$

where $A(z) = (A_{ij}(z))_{1\le i,j\le n}$ with each A_{ij} holo. on $\mathbb{C}\setminus\{z_1, \dots, z_k\}$

$$Y = (y_1 \cdots y_n)^T$$

Thm 1 For $\forall\, b \in \mathbb{C}\setminus\{z_1, \dots, z_k\}$, and $\forall\, v = (v_1, \dots v_n)^T \in \mathbb{C}^n$.
$\exists$ a small open $b \in U \subseteq \mathbb{C}\setminus\{z_1, \dots, z_k\}$ & $!$ solution Y_0 on U s.t. $Y_0(b) = v$.



Now fix b, and take a loop γ based at b.

$\rightsquigarrow$ solution $Y(z)$ & $Y(z\cdot \exp(2\pi i))$

Thm 1 $\Rightarrow$

$$Y(z) = Y(z\exp(2\pi i))\cdot g_\gamma \qquad \text{for } g_\gamma \in Gl_n\mathbb{C}$$

Moreover, if $\gamma_1 \sim \gamma_2 \Rightarrow g_{\gamma_1} = g_{\gamma_2}$

$\rightsquigarrow$ get monodromy representation

$$\rho : \pi_1(\mathbb{C}\setminus\{z_1, \dots, z_k\}, b) \longrightarrow Gl_n\mathbb{C}$$
$$[\gamma] \longmapsto g_\gamma$$

$\Leftrightarrow T_1, \dots, T_k \in Gl_n\mathbb{C}$

(since $\pi_1(\cdots, b)$ is a free gp. with k generators).

The problem:

Given rep. $\rho: \pi_1(\mathbb{C}\setminus\{z_1, \dots, z_k\}, b) \to GL_n\mathbb{C}$. can we find a system of linear ODEs of Fuchsian type so that the associated monodromy rep. is ρ?

Fuchsian: $\frac{dY}{dz} + \left(\sum_{i=1}^{k} \frac{A_i}{z-z_i}\right) Y = 0$

$A_i \in \mathfrak{gl}_n\mathbb{C} \quad 1 \le i \le k$

Rmk:

- Bolibrukh-Kostov: positive if ρ is irreducible, i.e. $\nexists$ non-trivial proper ρ-inv. subspace of $\mathbb{C}^n$
- Bolibrukh (1989): negative in general
  he constructed counterexample for $n=3=k$

In the following we will study similar problems related to it.

Goal: X complex analytic variety. i.e. complex mfd.
Then we have an equiv. of cats.:

$$\left\{ \begin{array}{c} \text{Local systems of} \\ \mathbb{C}\text{-v.s. on } X \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Holomorphic flat bundles} \\ (E, \nabla) \text{ on } X \end{array} \right\}$$

$$\left(\begin{array}{c} \text{if } X \text{ projective} \rightsquigarrow \ \wr\wr \ \text{GAGA} \\ \left\{ \begin{array}{c} \text{Algebraic flat bundles} \\ (E, \nabla) \text{ on } X^{alg} \end{array} \right\} \end{array} \right)$$

§2. Local systems & fundamental gp. reps.

Let X be a topological space (connected, locally simply-connected, Hausdorff, ...)

Def. A <u>local system</u> of $\mathbb{C}$-v.s. is a locally constant sheaf $\mathcal{F}$ of finite rank. i.e. $\mathcal{F}$ is locally iso. to the constant sheaf $\mathbb{C}^n_X$.

equiv. $\forall x \in X$, $\exists$ open $U \subseteq X$ s.t.

$$\mathcal{F}|_U \cong \mathbb{C}^n_U .$$

Lem: $X = [0,1]$ or $[0,1] \times [0,1]$, $\mathcal{F}, \mathcal{G}$ local systems on X. then

(1) If $s_0 \in \mathcal{F}_0$, then $\exists!$ section $s \in \Gamma(X, \mathcal{F})$ s.t. $s(0) = s_0$

(2) If $\varphi_0 : \mathcal{F}_0 \to \mathcal{G}_0$ a morphism, then $\exists!$ homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$

s.t. $\varphi|_{\mathcal{F}_0} = \varphi_0$.

((1) $\Rightarrow$ (2) since $\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}, \mathcal{G})$ is a local system)

$\cdot$ if φ_0 iso $\Rightarrow \varphi$ iso.

Now, let $\mathcal{F}$ be a local system on X, choose a base pt $x_0 \in X$ and take a loop $\gamma : [0,1] \to X$ based at x_0.

$\leadsto \gamma^{-1}\mathcal{F}$ is a local system on $[0,1]$

$[0,1]$ is simply-connected $\Rightarrow \gamma^{-1}\mathcal{F}$ is constant sheaf

Lem $\overset{(1)}{\Rightarrow}$

$$T_\gamma : \mathcal{F}_{x_0} = (\gamma^{-1}\mathcal{F})_0 \longrightarrow (\gamma^{-1}\mathcal{F})_1 = \mathcal{F}_{x_0}$$
$$s_0 \longmapsto s(1)$$

Lem (1) $\Rightarrow$ T_γ is <u>linear</u>, i.e. $T_\gamma(s_0 + \lambda v_0) = T_\gamma(s_0) + \lambda T_\gamma(v_0)$

$\lambda \in \mathbb{C}$

Prop: T_γ is homotopy-invariant, i.e. if $\gamma \sim \gamma' \Rightarrow T_\gamma = T_{\gamma'}$

Pf.

$\gamma \sim \gamma' \Rightarrow \exists$ cont. $H : [0,1] \times [0,1] \to X$ s.t.

$$H(0,t) = \gamma(t)$$
$$H(1,t) = \gamma'(t)$$

$\leadsto \gamma^{-1}\mathcal{F} = H^{-1}\mathcal{F}|_{\{0\} \times [0,1]}$ $\qquad \gamma'^{-1}\mathcal{F} = H^{-1}\mathcal{F}|_{\{1\} \times [0,1]}$

let $\varphi_0 : (\gamma^{-1}\mathcal{F})_0 \longrightarrow (\gamma'^{-1}\mathcal{F})_0$ be $s_0 \mapsto s_0$

$$\begin{tikzcd}
(H^{-1}\mathcal{F})_{(0,0)} \arrow[r, "T_\gamma"] \arrow[d, "\shortparallel"'] & (H^{-1}\mathcal{F})_{(0,1)} \arrow[d, "\shortparallel \leftarrow \text{lem (2)}"] \\
(H^{-1}\mathcal{F})_{(1,0)} \arrow[r, "T_{\gamma'}"] & (H^{-1}\mathcal{F})_{(1,1)}
\end{tikzcd}$$

$\Rightarrow \quad T_\gamma = T_{\gamma'}$

▨

In conclusion, if $\mathcal{F}$ local system (of $\mathbb{C}$-v.s. of rk n) on X

$\rightsquigarrow$

$$\begin{aligned} \rho_{\mathcal{F}} : \pi_1(X, x_0) &\longrightarrow GL_n\mathbb{C} \\ [\gamma] &\longmapsto T_\gamma \end{aligned}$$

$\rightsquigarrow$

a functor between cats.

$$\begin{aligned} \left\{ \begin{array}{l} \text{Local systems of} \\ \mathbb{C}\text{-v.s. of finite rk} \end{array} \right\} &\longrightarrow \left\{ \begin{array}{l} \text{Finite dim'l reps.} \\ \text{of } \pi_1(X, x_0) \end{array} \right\} \\ \mathcal{F} &\longmapsto \rho_{\mathcal{F}} \end{aligned}$$

Thm $\mathcal{F} \mapsto \rho_{\mathcal{F}}$ is an equiv. of cats.

§3. The correspondence

Now, X be a cplex analytic variety

$E \to X$ be a holo. vector bundle.

Recall. equiv. of cats.

$$\begin{aligned} \left\{ \begin{array}{l} \text{Holo. vector bundles} \\ \text{on } X \end{array} \right\} &\xrightarrow{\cong} \left\{ \begin{array}{l} \text{Locally free sheaves} \\ \text{of } \mathcal{O}_X\text{-modules} \end{array} \right\} \\ E &\longmapsto \mathcal{E} \end{aligned}$$

$\nwarrow$ sheaf of holo. sections of E

Let ∇ be a flat connection on E, i.e.

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$$

satisfying Leibniz rule & $\nabla^2 = 0$

proof of main thm.

"$\Leftarrow$": given (E, ∇) flat bundle. define

$\mathcal{E}^\nabla := \mathrm{Ker}(\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1) \subseteq \mathcal{E}$ subsheaf.

- $\mathcal{E}^\nabla$ is a local system:

$U \cap V \neq \phi$. let $\{s_i\}_{i=1}^n$ be a local frame of $\mathcal{E}^\nabla(U)$ ($n = \mathrm{rk}\, E$)

$\{s_i'\}_{i=1}^n$ ... $\mathcal{E}^\nabla(V)$

suppose $s_i' = \sum_{j=1}^n g_{ij} s_j$ for g_{ij} transition functions

Applying ∇ $\rightsquigarrow$ $0 = \nabla s_i' = \sum_{j=1}^n dg_{ij}\, s_j$ $\forall i$

$\Rightarrow dg_{ij} = 0 \quad \forall i, j.$

$\Rightarrow g_{ij}$ locally constant

"$\Rightarrow$" $\mathcal{F}$ local system. define $\mathcal{E} := \mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_X$. $\nabla = 1 \otimes d$

for $d : \mathcal{O}_X \to \Omega_X^1$

then $\nabla^2 = 0$ and $\mathcal{E}^\nabla = \mathcal{F} \otimes_{\mathbb{C}} \underline{\mathbb{C}} \simeq \mathcal{F}$

$\boxtimes$

Cor: We have the following equiv. of cats.

$$\left\{ \begin{array}{c} \text{Flat bundles on} \\ X \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Finite dim'l reps.} \\ \text{of } \pi_1(X, x_0) \end{array} \right\}$$

Rmk:

(E, ∇) flat bundle. take a base pt $x_0 \in X$

$\gamma : [0,1] \to X$ loop based at x_0.

locally around x_0, consider parallel section s with initial value

$s(\gamma(0)) = s_0 \in E_{x_0} \cong \mathbb{C}^n$

$\rightsquigarrow$

$\nabla s(\gamma(t)) = 0$

$s = \sum f_i s_i$ ← frame $\qquad \nabla s_i = \sum_j A_{ij} s_j$

$$\rightsquigarrow \nabla s(\gamma(t)) = 0 \iff \frac{d f(\gamma(t))}{dt} + A(\gamma(t)) f(\gamma(t)) = 0$$

By the previously mentioned ODE theory, obtain

$$\rho : \pi_1(X, x_0) \to GL(E_x) \cong GL_n\mathbb{C}$$

4. Some further remarks

For X complex projective variety. let X^{an} be the corresponding analytic variety.

Serre's GAGA $\Rightarrow$

$$\left\{ \begin{matrix} \text{Holo. flat bundles} \\ \text{on } X^{an} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Algebraic flat bundles} \\ \text{on } X \end{matrix} \right\}$$

Cor: X cplx prj.

$$\left\{ \begin{matrix} \text{Local systems of } \mathbb{C}\text{-v.s.} \\ \text{on } X \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Alg. flat bundles} \\ \text{on } X \end{matrix} \right\}$$

But, if X is a general algebraic (smooth) variety. i.e. punctured R.S.

GAGA does not hold!

Thm (Deligne) X smooth alg. variety $/\mathbb{C}$ or proper. then equiv. of cats

$$\left\{ \begin{matrix} \text{Holo. flat bundles} \\ \text{on } X^{an} \end{matrix} \right\} \xrightarrow{\cong} \left\{ \begin{matrix} \text{Regular singular flat} \\ \text{bundles on } X \end{matrix} \right\}$$

$$\wr\wr$$

$$\left\{ \begin{matrix} \text{Local systems} \\ \text{on } X \end{matrix} \right\}$$

(E, ∇) on X has an extension $\overline{E} \to \overline{X}$

and ∇ extends to $\overline{\nabla} : \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}}(\log D)$

$D = \overline{X} \setminus X$

Rmk:

If we consider irregular singular flat bundles, then left hand side should be

Stokes local systems (Deligne, Malgrange, Sibuya, Mochizuki, ...)

Thm: $M_{dR}(X,n) \cong M_B(X,n)$ cplx analytic isomorphism

pf idea:

fix $x \in X$

$R_{dR}(X,x,n)$ fine moduli space of framed flat bundles

$R_B(X,x,n) = \mathrm{Hom}(\pi_1(X,x), GL_n(\mathbb{C}))$

① $R_{dR}^{(an)}(X,x,n) \cong R_B^{(an)}(X,x,n)$

because $R_{dR}^{(an)}(X,x,n)$ & $R_B^{(an)}(X,x,n)$ represent the same moduli functors

②
$$\begin{tikzcd} R_{dR}^{(an)}(X,x,n) \arrow[r, "\cong"] \arrow[d, "/\!/GL_n\mathbb{C}"'] & R_B^{(an)}(X,x,n) \arrow[d, "/\!/GL_n(\mathbb{C})"] \\ M_{dR}^{(an)}(X,n) \arrow[r, "\cong"'] & M_B^{(an)}(X,n) \end{tikzcd}$$

$\boxtimes$