Projective GIT for linear actions

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The notes of this talk are based on [Hos15, Sections 5.1, 5.2, 6.1]. In this talk, we begin with the theory of projective GIT quotients in the case where G acts linear on a projective variety:

Setting:

- G will be a reductive affine algebraic group.
- $G \to \mathrm{GL}_{n+1}$ is a fixed group homomorphism, so G acts linearly on $\mathbb{CP}^n$.
- $X \subset \mathbb{CP}^n$ is a closed G-subvariety.

Under these circumstance we say that G acts linearly on $X \subset \mathbb{CP}^n$. Note: we really think of X as a subvariety of a given $\mathbb{CP}^n$, so the embedding is part of the data.

Let us denote by S the ring $\mathbb{C}[x_0, \ldots, x_n]$. We have seen that X is determined by

$$I(X) = \langle \{ f \in S \text{ homogeneous} \mid f(p) = 0 \quad \forall p \in X \} \rangle \tag{1}$$

We have that $R(X) := S/I(X)$ is a graded $\mathbb{C}$-algebra. In fact $\operatorname{Proj} R(X) \cong X$. For simplicity, we will assume as in the discussion session that X is irreducible, so $R(X)$ is integral. Furthermore $\operatorname{Spec} R(X) \cong \tilde{X} \subset \mathbb{C}^{n+1}$ is the affine cone of X. If $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the projection map, then $\tilde{X} = \pi^{-1}(X) \cup \{0\}$.

Now $G \circlearrowright R(X) = \bigoplus_{r \geq 0} R(X)_r$ preserves the grading. From this we obtain $R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G$ as a graded subalgebra of $R(X)$ with grading $(R(X)^G)_r = (R(X)_r)^G$. By Nagata's theorem, $R(X)^G$ is finitely generated. So the inclusion $R(X)^G \hookrightarrow R(X)$ gives a rational morphism:

$$\operatorname{Proj} R(X) \dashrightarrow \operatorname{Proj} R(X)^G \tag{2}$$

We describe the indeterminacy locus and the domain of definition of this morphism by the following definitions related to $R(X)_+^G = \bigoplus_{r>0} R(X)_r^G$:

Definition 1.

- A point $x \in X$ is called *unstable* if $f(x) = 0$ for all $f \in R(X)_+^G$ homogeneous. The set of all unstable points is called *null cone* N. (More accurately, $\pi^{-1}(N) \cup \{0\}$ the (affine) null cone, and N is the projective variety associated to it).
- A point $x \in X$ is called *semistable* if $f(x) \neq 0$ for some $f \in R(X)_+^G$ homogeneous. The set of all semistable points is called X^{ss}.

Then X^{ss} is the domain of definition of (2), the map $X^{ss} \to X /\!\!/ G := \operatorname{Proj} R(X)^G$ is called the projective GIT quotient of the linear action of G on X.

Theorem 2. *If G is reductive affine algebraic group acting linearly on $X \subset \mathbb{CP}^n$, then $\phi\colon X \to X /\!\!/ G$ is a good quotient of the G-action on X^{ss}, moreover $X /\!\!/ G$ is a projective variety.*

Recall [Hos15, Definition 3.27] that a morphism between varieties $\phi\colon X \to Y$ is a good quotient of a G-action if:

a) ϕ is G-invariant

b) ϕ is surjective.

c) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\phi^{-1}(U))^G$.

d) If $W \subset X$ is a G-invariant closed subset of X, its image $\phi(W)$ is closed in Y.

Type	{ Type }	Algebraic Definition	Geometric Criterion
unstable	N	$\forall f \in R(X)_+^G, f(x) = 0$	$0 \in \overline{G \cdot \tilde{x}}$
semistable	X^{ss}	$\exists f \in R(X)_+^G, f(x) \neq 0$	$0 \notin \overline{G \cdot \tilde{x}}$
polystable	X^{ps}	$G \cdot x \subset X^{ss}$ relatively closed	$G \cdot \tilde{x}$ closed (?)
stable	X^s	$G \cdot x \subset X^{ss}$ relatively closed and $\dim G_x = 0$	$G \cdot \tilde{x}$ closed and $\dim G_{\tilde{x}} = 0$

Table 1: Different types of points x for a linear action $G \circlearrowright X$ and their characterization on a lift $\tilde{x}$ in the affine cone $\tilde{X}$

e) If W_1 and W_2 are disjoint G-invariant closed subsets, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint.

f) ϕ is affine (preimages of every affine open is affine)

Proof of theorem 2. Let us denote $Y = X /\!/ G = \operatorname{Proj} R(X)^G$. Since Y is the Proj of a finitely generated (integral) graded $\mathbb{C}$-algebra, it is a projective variety. For $f \in R(X)_+^G$ homogeneous, the sets $Y_f = \{ y \in Y \mid f(y) \neq 0 \}$ form an basis for the Zariski topology on Y. Now $\phi^{-1}(Y_f) = X_f = \{ x \in X \mid f(x) \neq 0 \}$ and we have

$$\mathcal{O}(Y_f) = (R(X)^G)_{(f)} \cong (R(x)_{(f)})^G \cong \mathcal{O}(X_f)^G \tag{3}$$

so $\phi_f \colon X_f \to Y_f \cong \operatorname{Spec} \mathcal{O}(X_f)^G$ is an affine GIT quotient, hence good. By covering Y with the affine opens Y_f, we see that ϕ is a gluing of good quotients, so it is also good (see [Hos15, Remark 3.34]). □

In particular, we have have for $x_1, x_2 \in X^{ss}$ that

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset \Leftrightarrow \phi(x_1) = \phi(x_2) \tag{4}$$

The existence of non-closed orbits in X^{ss} prevent $X^{ss} \to X /\!/ G$ from being a geometric quotient. In order to obtain a geometric quotient we introduce the following set:

Definition 3. A point $x \in X$ is *stable* if we have the following:

- x is semistable
- $G \cdot x$ is closed in X^{ss}
- G_x is zero dimensional.

The set of stable points is denoted by X^s.

Note that this is not the definition given in [Hos15], but the equivalent description as [Hos15, Lemma 5.9]. With this we have the following:

Lemma 4. *X^s and X^{ss} are open subsets of X.*

Theorem 5 ([Hos15, Theorem 5.6])**.** *There is an open subvariety $Y^s \subset Y = X /\!/ G$ such that $\phi^{-1}(Y^s) = X^s$ and the GIT quotient restrict to a geometric quotient $\phi \colon X^s \to Y^s$.*

Definition 6. A point $x \in X$ is polystable if it is semistable and $G \cdot X \subset X^{ss}$ is relatively closed.

Lemma 7. *If $x \in X$ is semistable, then $\overline{G \cdot x}$ contains a unique polystable orbit.*

We can give the following topological criterion for stability:

Lemma 8. *Let $x \in X$, let $\tilde{x} \in \tilde{X}$ a lift of x (i.e. $\pi(\tilde{x}) = x$). We have the following:*

a) x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$

b) x is stable if and only if $G_{\tilde{x}}$ is zero dimensional and $G \cdot \tilde{x}$ is closed in $\tilde{x}$.

We organize this information in table 1 (adapted from [Tho23, Proposition 6.5]).

Example 1. *Consider the action of $\mathbb{C}^*$ on $\mathbb{CP}^2$ given by the homomorphism*

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \tag{5}$$

Using the geometric criterion we see that $N = \{ z = 0 \} \cup \{ [0:0:1] \}$, and $X^s = X^{ss} = \mathbb{CP}^2 \setminus N$. In this example, every semistable point is in fact stable. If we denote the homogeneous coordinate ring of $X = \mathbb{CP}^2$ by $\mathbb{C}[x_0, \ldots, x_2]$, then $R(X)^G = \mathbb{C}[x_0x_2, x_1x_2]$, $X /\!\!/ G = \operatorname{Proj} R(X)^G \cong \mathbb{CP}^1$, with quotient map

$$[x_0 : x_1 : x_2] \mapsto [x_0x_2 : x_1x_2] \tag{6}$$

Example 2. *Consider the action of $\mathbb{C}^*$ on $\mathbb{CP}^2$ given by the homomorphism*

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \tag{7}$$

Using the geometric criterion we see that $N = \{ [1:0:0] \cup [0:0:1] \}$, and that $[0:1:0]$ is polystable but not stable. We have that $[0:1:0]$ is the unique polystable point in the orbit closures of $\{ z = 0 \}$ and $\{ x = 0 \}$. So $X^s = \mathbb{CP}^2 \setminus (\{ z = 0 \} \cup \{ x = 0 \})$, $X^{ps} = X^s \cup [0:1:0]$. In particular, we have $X^s \subsetneq X^{ps} \subsetneq X^{ss} \subsetneq X$. We have $R(X)^G = \mathbb{C}[x_0x_2, x_1]$. If we consider the Veronese subring $(R(X)^G)^(2) = \mathbb{C}[x_0x_2, x_1^2]$, we see again that $X /\!\!/ G \cong \mathbb{CP}^1$. In fact, the description of $R(X)^G = \mathbb{C}[x_0x_2, x_1]$ gives us that $X /\!\!/ G$ can be thought of a weighted projective space $\mathbb{P}(1,2)$, but in dimension 1, they are all isomorphic to $\mathbb{CP}^1$. For subtleties related to weighted projective spaces see [Dol82, Section 1.5].

Two extremal examples:

Example 3. *If G acts trivially, then $X /\!\!/ G \cong X$, but if G is positive dimensional, then X^s is empty.*

Example 4. *Consider the action of $\mathbb{C}^*$ on $\mathbb{CP}^2$ given by the homomorphism*

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \tag{8}$$

Then every point is unstable, so $N = \mathbb{CP}^2$, and $X /\!\!/ G = \emptyset$. These examples also show that the datum of the group homomorphism $G \to \mathrm{GL}^{n+1}$ can really affect the resulting quotient.

References

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[Hos15] Victoria Hoskins. Moduli problems and geometric invariant theory. *Available at `https://userpage.fu-berlin.de/hoskins/M15_Lecture_notes.pdf`*, 2015.

[Tho23] Alexander Thomas. A gentle introduction to the non-abelian hodge correspondence. *Available at `https://arxiv.org/abs/2208.05940`*, 2023.