HILBERT–MUMFORD CRITERION

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Abstract. In this talk we introduce the Hilbert–Mumford criterion, a numerical criterion simplifying checking for (semi-)stability. The main reference is [2, §6].

Setup. Throughout this talk $X \subseteq \mathbb{P}^n$ denotes a projective variety, equipped with a G-linear action, where G is a reductive group. Recall that this means that G acts on $\mathbb{P}^n$ via a group homomorphism $G \to \mathrm{GL}_{n+1}$ and X is a closed G-subvariety of $\mathbb{P}^n$.

Definition 1. A *1-parameter subgroup* λ of G is a nontrivial group homomorphism $\lambda\colon \mathbb{C}^* \to G$.

For a point $x \in X$ we then define $\lambda_x\colon \mathbb{C}^* \to X$ by $\lambda_x(t) \coloneqq \lambda(t).x$. As the torus includes in $\mathbb{C}^* \hookrightarrow \mathbb{P}^1$, we can define the limits

$$\lim_{t\to 0} \lambda_x(t) \quad \text{and} \quad \lim_{t\to\infty} \lambda_x(t) = \lim_{t\to 0} \lambda_x^{-1}(t).$$

These are well-defined by compactness of $\mathbb{P}^1$ (or, as the algebraic geometer likes to put it, by the valuative criterion for properness).

Let $\tilde{X} \subseteq \mathbb{A}^{n+1}$ be the affine cone over X and let $\tilde{x} \in \tilde{X} \setminus \{0\}$ be a lift of x. As above, we may define $\lambda_{\tilde{x}}\colon \mathbb{C}^* \to \tilde{X},\ \lambda_{\tilde{x}}(t) \coloneqq \lambda(t).\tilde{x}$; however, the limits might not exist.

Fact 2 ([2, Prop. 3.12])**.** The action of $\lambda(\mathbb{C}^*)$ on $\mathbb{A}^{n+1}$ is *diagonalisable*, i.e. there exists a basis $(e_0, e_1, \dots, e_n)$ of $\mathbb{C}^{n+1}$ and integers $r_0, r_1, \dots, r_n \in \mathbb{Z}$ such that

$$\forall t \in \mathbb{C}^*\colon \lambda_{e_i}(t) = t^{r_i} e_i.$$

Let us write $\tilde{x} = \sum_{i=0}^n x_i e_i$; then $\lambda_{\tilde{x}}(t) = \sum_{i=0}^n t^{r_i} x_i e_i$.

Definition 3. The *Hilbert–Mumford weight* of x at λ is

$$\mu(x, \lambda) \coloneqq -\min\{r_i\colon x_i \neq 0\}.$$

Note that this definition does not depend on the choice of the lift $\tilde{x}$. We collect some important properties below.

Proposition 4.

(1) $\mu(x,\lambda)$ is the unique integer $\mu \in \mathbb{Z}$ such that $\lim_{t\to 0} t^{\mu}\lambda_x(t)$ exists and is non-zero
(2) for all $n \in \mathbb{N}$, $\mu(x, \lambda^n) = n\mu(x,\lambda)$
(3) for all $g \in G$, $\mu(g.x, g\lambda g^{-1}) = \mu(x,\lambda)$
(4) $\mu(x,\lambda) = \mu(y,\lambda)$ where $y = \lim_{t\to 0} \lambda_x(t)$.

Proof. We have $\lim_{t\to 0} t^{\mu(x,\lambda)}\lambda_x(t) = \lim_{t\to 0} t^{\mu(x,\lambda)} \sum_{i=0}^n t^{r_i} x_i e_i = \sum_{j\colon r_j = -\mu(x,\lambda)} x_j e_j$ which shows part (1). The other properties are immediate consequences of (1). ☐

As a consequence, we obtain the following.

Proposition 5.

(1) $\mu(x,\lambda) < 0$ if and only if $\lim_{t\to 0} \lambda_{\tilde{x}}(t) = 0$
(2) $\mu(x,\lambda) = 0$ if and only if $\lim_{t\to 0} \lambda_{\tilde{x}}(t)$ exists and is nonzero
(3) $\mu(x,\lambda) > 0$ if and only if $\lim_{t\to 0} \lambda_{\tilde{x}}(t)$ does not exist.

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type	geometric	Hilbert–Mumford
unstable	$0 \in \overline{G.\tilde{x}}$	$\exists$ 1-ps λ: $\mu(x,\lambda) < 0$
semistable	$0 \notin \overline{G.\tilde{x}}$	$\forall$ 1-ps λ: $\mu(x,\lambda) \geq 0$
stable	$G.\tilde{x} = \overline{G.\tilde{x}}$ & $\dim G_{\tilde{x}} = 0$	$\forall$ 1-ps λ: $\mu(x,\lambda) > 0$

Table 1. Criteria for stability

Theorem 6 (Hilbert–Mumford Criterion)**.**

(1) $x \in X^{ss}$ if and only if $\mu(x,\lambda) \geq 0$ for all 1-parameter subgroups λ of G
(2) $x \in X^{s}$ if and only if $\mu(x,\lambda) > 0$ for all 1-parameter subgroups λ of G.

Proof. We will show the "only if" direction; the other direction is more involved. The interested reader is referred to [2] or [3, §7] for a proof in the case $G = GL_{n+1}$.

Let x be semistable. We have seen in the previos talk that this is equivalent to $0 \notin \overline{G.\tilde{x}}$ (see Table 1). The limit of any 1-parameter subgroup (if it exists) will be contained in the orbit closure of x. By Proposition 5, this implies $\mu(x,\lambda) \geq 0$.

Now let x be stable. This is equivalent to the G-orbit being closed, $G.\tilde{x} = \overline{G.\tilde{x}}$, and the stabiliser $G_{\tilde{x}}$ being finite. Let λ be a 1-parameter subgroup of G and assume per contradiction that $\lim_{t\to 0} \lambda_{\tilde{x}}(t)$ exists; let y denote this limit. As $G.\tilde{x} = \overline{G.\tilde{x}}$, y is contained in the orbit $G.\tilde{x}$. Then also y must have finite stabiliser. However, $\lambda(t)$ stabilises y for any $t \in \mathbb{C}^*$:

$$\lambda(t).y = \lambda(t) \lim_{s\to 0} \lambda(s).\tilde{x} = \lim_{s\to 0} \lambda(st).\tilde{x} = y.$$

Hence, $G_{\tilde{x}}$ is at least 1-dimensional, a contradiction. Therefore, the limit $\lim_{t\to 0} \lambda_{\tilde{x}}(t)$ does not exist which is, by Proposition 5, equivalent to $\mu(x,\lambda) > 0$. ☐

The following statement is equivalent to the Hilbert–Mumford Criterion and is called "Fundamental Theorem of GIT".

Theorem 7. *Let G be a reductive group acting on $\mathbb{A}^{n+1}$. If $x \in \mathbb{A}^{n+1}$ is a closed point and $y \in \overline{G.x}$, then there exists a 1-parameter subgroup λ of G such that $\lim_{t\to 0} \lambda_x(t) = y$.*

Let us now take a look at two examples.

Example 8. (1) Let $X = \mathbb{P}^2$ and let $G = \mathbb{C}^*$ act on X via

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}.$$

All 1-parameter subgroups of G are of the form $t \mapsto t^m$ for $m \in \mathbb{Z}$. By the second property in Proposition 4, it is enough to check (semi-)stability only for $\lambda(t) = t$ and $\lambda^{-1}(t) = t^{-1}$. Let $x = [x_0 : x_1 : x_2]$ and choose $\tilde{x} = (x_0, x_1, x_2)$ as a lift.

Then $\lim_{t\to 0} \lambda_{\tilde{x}}\lambda(t) = \lim_{t\to 0}(tx_0, tx_1, t^{-1}x_2)$ exists if and only if $x_2 = 0$. If $x_2 = 0$ then $\mu(x,\lambda) = -1$, otherwise $\mu(x,\lambda) = 1$.

Similarly, $\lim_{t\to 0} \lambda_{\tilde{x}}^{-1}(t) = \lim_{t\to 0}(t^{-1}x_0, t^{-1}x_1, tx_2)$ exists if and only if $x = [0 : 0 : 1]$; in this case, $\mu(x,\lambda^{-1}) = -1$, otherwise $\mu(x,\lambda^{-1}) = 1$.

Hence we see $X^{ss} = X^{s} = \mathbb{P}^2 \setminus (\{x_2 = 0\} \cup \{[0 : 0 : 1]\})$.

(2) Let $G = SL_2$ act on $\mathbb{P}^1$. Every 1-parameter subgroup of SL_2 is conjugate to

$$\lambda_r : t \mapsto \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$$

for $r \in \mathbb{N}$. By properties (2) and (3) of Proposition 4, it is therefore enough to check semistability for $\lambda := \lambda_1$.

The action on $\mathbb{P}^1$ extends to an action on $\mathbb{P}(\mathrm{Sym}(2,d)) \cong \mathbb{P}^d$, the space of binary d-forms as follows: let $f(x,y) = a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d \in \mathbb{P}(\mathrm{Sym}(2,d))$; then

$$\lambda(t).f(x,y) = a_0 t^d x^d + a_1 t^{d-1} x^{d-1} y + \cdots + a_d t^{-d} y^d.$$

Claim 9. A binary form of degree d is semistable if and only if all roots have multiplicity at most $d/2$. It is stable if and only if all roots multiplicity less than $d/2$.

Proof. Assume that, possibly after a coordinate change, f has root $[0:1]$ with multiplicity $m > d/2$. Therefore, we can write $f(x,y) = \sum_{i \leq d-m} a_i x^{d-i} y^i$, and

$$\lim_{t \to 0} \lambda(t).f(x,y) = \lim_{t \to 0} \sum_{i \geq m} a_i t^{d-2i} x^{d-i} y^i.$$

Since $m > d/2$, all exponents of t appearing in the sum above are positive and the limit is zero, hence $\mu(f,\lambda) < 0$ and f is unstable. On the other hand, if $m < d/2$, some of the exponents of t are negative and the limit does not exist, and for $m = d/2$ the limit will be $a_{d/2} x^{d/2} y^{d/2}$. Note that in particular for odd d, the semistable and stable locus agree. □

Finally, let us mention the Kempf–Ness Theorem which relates the GIT quotient with symplectic reduction. See [1] for more details.

Theorem 10 (Kempf–Ness)**.** *Let $X \subseteq \mathbb{P}^n$ be a smooth complex variety, let G be a complex reductive group acting linearly on X, and let $K \subset G$ be a maximal compact subgroup of G. Then there is a homeomorphism*

$$X /\!\!/_{GIT}\, G \cong X /\!\!/_{symp}\, K = \mu^{-1}(0)/K,$$

where μ is the moment map associated to the action of K. In this case, $X^{ps} = G.\mu^{-1}(0)$.

References

[1] Victoria Hoskins. The Kempf–Ness theorem. Online Lecture Notes.
[2] Victoria Hoskins. Moduli problems and geometric invariant theory, 2015. Online Lecture Notes.
[3] Shigeru Mukai. *An introduction to invariants and moduli*, volume 81. Cambridge University Press, 2003. available online.