The Moduli problem

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A *naive moduli problem* (over $\mathbb{C}$) is just a collection of "interesting" objects from algebraic geometry $\mathcal{A}$ together with an equivalence relation $\sim$ on this collection. We would like to have an algebraic variety whose points parameterize the equivalence classes of $\mathcal{A}/\!\sim$ in such a way that nicely varying families of objects correspond to paths in this variety. We first make this precise.

1 Moduli problems

Definition 1. An *(extended) moduli problem* is a collection $\mathcal{A}(S)$ equipped with an equivalence relation $\sim_S$ for every variety S, together with pullback map $f^*\colon \mathcal{A}(T) \to \mathcal{A}(S)$ for all morphisms $f\colon S \to T$. The pullback maps must obey the following:

- $(\mathrm{id}_S)^* = \mathrm{id}_{\mathcal{A}(S)}$ for all S, so $\mathrm{id}_S^*(F) = F$ for $F \in \mathcal{A}(S)$;
- if $E \sim_T F$ in $\mathcal{A}(T)$ and $f\colon S \to T$, then $f^*E \sim_S f^*F$;
- if $E \in \mathcal{A}(T)$ and $R \xrightarrow{f} S \xrightarrow{g} T$, then $(g \circ f)^*E \sim_R g^*f^*E$ (often one even has equality);
- if $* = \operatorname{Spec}\mathbb{C}$ is a point, then $(\mathcal{A}(*), \sim_*)$ is the naive moduli problem we care about.

The *moduli functor* associated to a moduli problem is the contravariant functor

$$\mathcal{M}\colon \mathbf{Var}_{\mathbb{C}} \to \mathbf{Set}, \qquad S \mapsto \mathcal{A}(S)/\!\sim_S.$$

Example 2. The following are moduli problems, $\sim_S$ being equality if not specified. What are the naive moduli problems?

1. For $n \in \mathbb{N}_0$ the constant functor $\underline{[n]}(S) \coloneqq \{1, 2, \dots, n\}$.
2. For a fixed variety M the *functor of points* $h_X(S) \coloneqq \mathrm{Mor}_{\mathbb{C}}(S, M)$. Note that $h_M(*) = M$, hence the name.
3. The functor of global functions $\mathbb{G}_{\mathrm{a}}(S) \coloneqq \mathcal{O}_S(S)$, or the functor of invertible functions $\mathbb{G}_{\mathrm{m}}(S) \coloneqq \mathcal{O}_S(S)^\times$.
4. For $0 \le k \le n$ we have the *Grassmannian functor*

   $$\mathcal{G}r_{k,n}(S) \coloneqq \{\, q\colon S \times \mathbb{C}^n \twoheadrightarrow \mathcal{E} \mid \mathcal{E} \text{ is a quotient vector bundle of rank } n-k \,\}$$

   where $\sim_S$ is isomorphism of vector bundles $f\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ with $f \circ q = q'$.
5. For a fixed projective variety $X \in \mathbf{Var}_{\mathbb{C}}$ the *Hilbert functor*

   $$\mathcal{H}ilb_X(S) = \{\, Z \subseteq X \times S \mid Z \text{ a closed subscheme flat over } S \,\}$$

   with equality (or, pedantically, isomorphism as closed embeddings $Z \hookrightarrow X \times S$). Here "flat" informally means that the fibers of $Z \to S$ vary nicely.

6. The *Moduli functor of smooth curves of genus g* is

$$\mathcal{M}_g(S) = \left\{ \pi\colon X \to S \;\middle|\; \begin{array}{c} \pi \text{ is a smooth morphism and for } s \in S \text{ the fiber} \\ \pi^{-1}(s) \text{ is a conn. smooth proj. curve of genus } g. \end{array} \right\}$$

   The equivalence relation is as isomorphism of varieties over S. We could also drop g and consider all relative smooth projective curves at once.

7. As a variant of the previous, we have the *moduli functor of marked smooth curves*

$$\mathcal{M}_{g,n}(S) = \{ (\pi, p_1, \dots, p_n) \mid \pi \text{ as before and } p_1, \dots, p_n \text{ are disjoint sections of } \pi \}$$

8. Let π be a finitely presented group and G an algebraic reductive group.

$$\mathfrak{R}(\pi, G)(S) = \{ ? \}$$

2 Coarse and fine moduli spaces

To define the notion of a moduli space, we need to recall the definition of a natural transformation. Let $\mathcal{F}, \mathcal{G}\colon \mathbf{Var}_{\mathbb{C}} \to \mathbf{Set}$ be contravariant functors. A natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a collection of maps $\eta_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ such that for all $f\colon X \to Y$ one has

$$\eta_X \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_Y \qquad \in \mathrm{Mor}_{\mathbf{Set}}(\mathcal{F}(Y), \mathcal{G}(X)).$$

A natural isomorphism is a natural transformation which has an inverse, equivalently all η_X are isomorphisms (here in **Set**: bijective).

Lemma 3 (Yoneda)**.** *The functors of points provide a fully faithful embedding*

$$h_{(-)}\colon \mathbf{Var}_{\mathbb{C}} \hookrightarrow \mathrm{Fun}(\mathbf{Var}_{\mathbb{C}}^{\mathrm{opp}}, \mathbf{Set}), \qquad M \mapsto h_M.$$

This means that $h_{(-)}$ is a covariant injective functor from $\mathbf{Var}_{\mathbb{C}}$ to presheaves such that $h\colon \mathrm{Mor}_{\mathbb{C}}(X,Y) \to \mathrm{Nat}(h_X, h_Y)$ is a bijection. Even more is true:

For any contravariant functor $\mathcal{F}\colon \mathbf{Var}_{\mathbb{C}}^{\mathrm{opp}} \to \mathbf{Set}$ one has

$$\begin{aligned} \mathrm{Nat}(h_M, \mathcal{F}) &\leftrightarrow \mathcal{F}(M) \\ \eta &\mapsto \eta_M(\mathrm{id}_M) \\ \eta(x) &\leftarrow\!\shortmid x, \qquad \eta(x)_X\colon \mathrm{Mor}(X, M) \ni f \mapsto \mathcal{F}(f)(x) \in \mathcal{F}(X). \end{aligned}$$

Definition 4. Let $\mathcal{M}$ be a moduli functor associated to a moduli problem, M a variety and $\eta\colon \mathcal{M} \Rightarrow h_M$ a natural transformation.

1. (M, η) is a *fine moduli space* for $\mathcal{M}$ if η is a natural isomorphism.

2. (M, η) is a *coarse moduli space* if $\eta_{\mathrm{Spec}\,\mathbb{C}}\colon \mathcal{M}(\mathrm{Spec}\,\mathbb{C}) \to M$ is bijective and (M, η) is initial among pairs (N, ε) of varieties with natural transformations from $\mathcal{M}$ to their functor of points.

Initial means here: For any variety N and any transformation $\varepsilon\colon \mathcal{M} \Rightarrow h_N$ there exists a *unique* morphism $f\colon M \to N$, equivalently by Yoneda, a unique transformation $h_f\colon h_M \Rightarrow h_N$, such that $\varepsilon = h_f \circ \eta$.

Lemma 5. *1. Coarse moduli spaces are uniquely unique: If (M', η') is another coarse moduli space, then there is a unique isomorphism of varieties $f\colon M \xrightarrow{\sim} M'$ such that $\eta' = h_f \circ \eta$.*

2. *Fine moduli spaces are coarse moduli spaces (and hence also uniquely unique).*

Proof. The first part follows from the universal property of coarse moduli spaces. For the second part: A natural isomorphism $\mathcal{M} \leftrightarrow h_M$ induces a bijection $\mathcal{M}(\operatorname{Spec}\mathbb{C}) \leftrightarrow M$. Furthermore, if (N, ε) is another tuple, then choose $h_f := \varepsilon \circ \eta^{-1} \colon h_M \Rightarrow h_N$; this choice is forced upon us by the equation $\varepsilon = h_f \circ \eta$. □

The set $h_M(M)$ has the distinguished element id_M. If (M, η) is a fine moduli space, then this determines a distinguished element in $\mathcal{M}(M)$.

Definition 6. For a fine moduli problem (M, η) the element $\mathcal{U} := \eta_M^{-1}(\mathrm{id}_M) \in \mathcal{M}(M)$ is called the *universal family*.

The following lemma justifies the name by showing that *every* family (!) is a pullback of the universal family.

Lemma 7. *Let $F \in \mathcal{A}(S)$ be a family over S, then $f := \eta_S([F]_{\sim_S})$ satisfies $f^*\mathcal{U} \sim_S F$. In other words: If $F \in \mathcal{M}(S)$, then $\mathcal{M}(\eta_S(F))(\mathcal{U}) = F$.*

Proof. The second claim is equivalent to the first by the construction of $\mathcal{M}$ from $\mathcal{A}$ and $\sim$. The second claim follows from the definition of a natural transformation: For $f \colon S \to M$ we have $\eta_S \circ \mathcal{M}(f) = h_M(f) \circ \eta_M$, plugging in $\mathcal{U}$ yields

$$\begin{aligned}\mathcal{M}(f)(\mathcal{U}) &= \eta_S^{-1} \circ \eta_S \circ \mathcal{M}(f)(\mathcal{U}) = \eta_S^{-1} \circ h_M(f) \circ \eta_M(\mathcal{U}) \\ &\overset{\text{def}\,\mathcal{U}}{=} \eta_S^{-1} \circ h_M(f) \circ \eta_M \circ \eta_M^{-1}(\mathrm{id}_M) = \eta_S^{-1}(f) =: F.\end{aligned}$$ □

In fact, for a pair (M, η) we *define* a universal family $\mathcal{U}$ to be an element $\in \mathcal{M}(M)$ such that for every $F \in \mathcal{M}(S)$ one has $F = \mathcal{M}(\eta_S(F))(\mathcal{U})$. The existence of a universal family somewhat characterizes fine moduli spaces, as the following lemma shows.

Lemma 8. *For a coarse moduli space (M, η) the following are equivalent:*

1. *(M, η) is a fine moduli space;*
2. *There is a $\mathcal{U} \in \mathcal{M}(M)$ with $\eta(\mathcal{U}) = \mathrm{id}_M$ and all η_S are injective.*

Example 9. The Grassmannian functors are representable by our beloved $\mathrm{Gr}(k, n) = \mathrm{Mat}(k \times n, \mathbb{C}) / \mathrm{GL}(k, \mathbb{C})$. The universal family is the quotient of the trivial bundle

$$0 \to \{\, (q, v) \in \mathcal{G}r_{k,n}(\mathbb{C}) \times \mathbb{C}^n \mid v \in \ker(q) \,\} \to \mathrm{Gr}(k, n) \times \mathbb{C}^n \to \mathcal{U} \to 0.$$

The left subbundle $\mathcal{T}$ is the *tautological bundle*; identifying $q \colon \mathbb{C}^n \to \mathbb{C}^{n-k}$, with its kernel, we have

$$\mathcal{T} \cong \{\, ([W], v) \mid W \subseteq \mathbb{C}^n \text{ of dim. k, } v \in W \,\}.$$

The Grassmannian is a fine moduli space, it is a smooth projective variety.

3 How to make a moduli space?

Often, moduli problems don't admit a fine moduli space, due to automorphisms. There are at least three options here:

- **Weaken the notion of moduli space:** Look for a coarse moduli space instead. For example, the functor $S \mapsto \mathrm{Pic}(S) = \{\, \text{line bundles on } S \,\} / \cong$ does not admit a fine moduli space, but $\{*\}$ is a coarse moduli space (why?).

- **Weaken the type of space:** The Hilbert functor is not representable as a fine moduli space in varieties. It is however representable in the category of projective schemes (after fixing the Hilbert polynomial). Similarly, the moduli space of curves is only a coarse moduli space (and only for $2g-2+n>0$), for example $M_{1,1}$, the space of (marked) elliptic curves, is $\mathbb{A}^1$, parametrized by the j-invariant. But it is always represented as a fine moduli space by a *Deligne-Mumford stack*, a category vastly extending the category of varieties.

- **Add data to the moduli problem:** The coarse moduli space of curves $M_{g,n}$ is actually a fine moduli space on the set of marked smooth curves with trivial automorphism group (these are dense if $2g-2+n>0$). Increasing n one can make all smooth curves automorphism-free, hence turn $M_{g,n}$ into a fine moduli space.

We now sketch how GIT enters the picture.

- Hilbert scheme can be constructed using Gotzmann number as a closed subscheme of $\mathrm{Gr}(k, I_G)$

- Every curve X of genus $g \geq 2$ has the property that $\Omega_{X,\mathbb{C}}^{\otimes 3}$ is a very ample line bundle ($\omega_{X,\mathbb{C}}$ is already ample for non-hyperelliptic curves).

- This gives closed embeddings $X \hookrightarrow \mathbb{P}(\Gamma(\Omega_{X,\mathbb{C}}^{\otimes 3})^*) = \mathbb{P}^{5g-6}$ of degree $6g-6$.

- The subvarieties of $\mathbb{P}^{5g-6}$ which are smooth connected curves of genus g and degree $6g-6$ form a locally closed subscheme K_g of the Hilbert scheme

- Two curves are isomorphic if and only if they are the same up to the action of $G = \mathrm{SL}(5g-7)$; this action induces an action on the Hilbert scheme.

- Take the GIT quotient (projective GIT in the stable curve case) $M_g = K_g//G$