Differential Geometric Construction of the Moduli Space of Higgs Bundles

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The talk is based on [AN16]. We are going to construct the moduli space of Higgs bundles over a compact Riemann surface using the space of doubled connections. This is a infinite dimensional hyperkähler space with a group action of the unitary gauge group, which admits a hyperkähler moment map. The moduli space of Higgs bundles is the hyperkähler quotient together with a specific complex structure. Under certain stability conditions, this will be a smooth Kähler manifold of finite dimension. As an example we consider it for line bundles of degree zero.

0. PRELIMINARIES

Throughout this whole talk X will denote a compact Riemann surface of genus $g \geq 2$ and $E \to X$ a complex vector bundle of degree d and rank n. Moreover, ω_X denotes a compatible Kähler form on X such that $1 = \int_X \omega_X$. For a fixed hermitian metric h on E we extend the Hodge-star to $\Omega^1(\mathrm{End}E)$ by defining it on products $\star(\omega A) := (\star\omega A^*)$ for $\omega \in \Omega^1(X)$ and $A \in \Omega^0(\mathrm{End}E)$, where A^* is the adjoint of A with respect to h. Then $\star^2 = -1$ and $\phi \wedge \star\psi = -\star\phi \wedge \psi$. Since we are going to take a qoutient of an infinte dimensional manifold by an infinite dimensional Lie group, we need to redefine Riemannian and hyperkähler structures.

Definition 0.1. *Let $\mathcal{M}$ be a possibly infinite dimensional smooth manifold and $p \in \mathcal{M}$.*

*A **Riemannian metric** g on $\mathcal{M}$ is a smoothly varing inner product g_p on the tangent spaces such that $T_p\mathcal{M} \ni X \mapsto g(X, \cdot) \in T_p\mathcal{M}^*$ is injective.*

*A Riemannian metric g together with three complex structures I_1, I_2, I_3 (defined as in finite dimensions) satisfying $I_i^2 = I_1 I_2 I_3 = -1$ is called a **hyperkähler structure** if the forms $\omega_i = g(I_i\,\cdot\,, \cdot\,)$ are closed and g is hermitian with respect to all complex structures.*

We will use the following theorem to prove the smoothness and the existence of a Hyperkähler structure on the Moduli space of Higgs bundles.

Theorem 0.2. *Let $\mathcal{G}$ be a possibly infinite dimensional Lie group acting freely and proper on a hyperkähler manifold $\mathcal{M}$ such that there exists a **hyperkähler moment map**, i.e. $\mu\colon \mathcal{M} \to (\mathfrak{g}^*)^{\oplus 3}$, satisfying $d_p\mu_i(Z) = \iota_{\rho(Z)}\omega_i$ for all $Z \in \mathfrak{g}, p \in \mathcal{M}$ and $i = 1, 2, 3$. If*

$$T_p(\mathcal{G}.p) \oplus T_p(\mathcal{G}.p)^\perp = T_p\mu^{-1}(0)$$

for all $p \in \mu^{-1}(0)$, then $\mu^{-1}(0)/\mathcal{G}$ is a smooth hyperkähler manifold.

Proof. In [AT07, p. 171, Theorem 2.22]. □

1. HIGGS BUNDLES

Definition 1.1. *A tuple $(E, \bar\partial, \varphi)$ consisting of a holomorphic vector bundle and a **Higgs field** $\varphi \in H^0(X, \mathrm{End}E \otimes K)$ is called a **Higgs bundle**. We consider $H^0(X, \mathrm{End}E \otimes K) \subset \Omega^{1,0}(\mathrm{End}E)$*

Given a Higgs Bundle $(E, \bar\partial, \varphi)$ a **Higgs subbundle** is a holomorphic subbundle $F \subset E$ that is φ-invariant, i.e. $\varphi(F) \subset F \otimes K$. The **slope** of a vector bundle $E \to X$ is given by the quotient $\mu(E) = \frac{\deg E}{\mathrm{rk}E}$. This topological quantity is used to define stability conditions for holomorphic vector bundles. In a similar manner, we define the notion of stability for Higgs bundles.

Definition 1.2. *A Higgs bundle $(E, \bar\partial, \varphi)$ is called*

*(i) **stable** if the slope of all proper Higgs subbundles is less then $\mu(E)$.*

*(ii) **polystable** if E decomposes as the direct sum of stable Higgs bundles E_i such that $\mu(E) = \mu(E_i)$ for all i.*

Let us look at a simple example of a stable Higgs bundle.

Example 1.1. *Consider a spin structure on X, i.e. a holomorphic line bundle $L \to X$ such that $L^2 \cong K$. Set $E = L \oplus L^{-1}$ and define a section of $\mathrm{End}E \otimes K$ by*

$$\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $\mathrm{Hom}(L, L^{-1})K = L^{-1}L^{-1}K = K^{-1}K = \underline{\mathbb{C}}_X$, φ is well defined. $1 \in H^0(\underline{\mathbb{C}}_X)$, thus (E, φ) is a Higgs bundle of rank 2 and degree 0. L and L^{-1} are the only proper holomorphic subbundles. We obtain $\varphi(L) = L^{-1}K \not\subset LK$ and $\varphi(L^{-1}) = 0 \subset L^{-1}K$, meaning that L^{-1} is the only proper Higgs subbundle. Further $\deg L^{-1} = -\frac{1}{2}\deg L = 1 - g < 0$. Hence, the Higgs bundle is stable.

We want to define the Moduli of Higgs bundles for fixed rank and degree. Since all vector bundles of same degree and rank over X are topologically equivalent we fix $E \to X$ and consider every Higgs bundle of degree d and rank n to be a pair $(\bar\partial, \varphi) \in \mathcal{A}_E^{\bar\partial} \times \Omega^{1,0}(\mathrm{EndE}) =:$

$\mathcal{A}_1^{\mathbb{C}}$ such that $\bar\partial\varphi = 0$. $\mathcal{A}_1^{\mathbb{C}}$ is modeled on the complex vector space $\Omega^{0,1}(\mathrm{End}E) \oplus \Omega^{1,0}(\mathrm{End}E)$. Moreover, the complex gauge group $\mathcal{G}_{\mathbb{C}} := \Omega^0(\mathrm{Aut}(E))$ acts on $\mathcal{A}_1^{\mathbb{C}}$ via $g.(\bar\partial, \varphi) = (g \circ \bar\partial \circ g^{-1}, g\varphi g^{-1})$. Thus, two Higgs bundles are in the same orbit if there exists a vector bundle isomorphism $g\colon E \to E$ that commutes with the holomorphic structures and the Higgs fields. Such a g is called an **isomorphism of Higgs bundles**.

$\square$

2. THE SPACE OF DOUBLED CONNECTIONS

Let h be a hermitian metric on E. We consider the unitary gauge group $\mathcal{G} := \Omega^0(UE) := \{g \in \Omega^0(\mathrm{Aut}E) \mid g^* = g^{-1}\}$ and its action on a space isomorphic to $\mathcal{A}_1^{\mathbb{C}}$. Locally the Lie group $\mathcal{G}$ is given by smooth maps $X \supset U \to U(n)$ with pointwise matrix multiplication. Thus, $\mathfrak{g} = \Omega^0(\mathfrak{u}E) := \{Z \in \Omega^1(\mathrm{End}E) \mid -Z = Z^*\}$ and we will use the identification

$$\Omega^2(\mathfrak{u}E) \cong \mathfrak{g}^*, \;\; F \mapsto \int_X \mathrm{tr}(F \wedge \cdot).$$

Definition 2.1. *A pair $(D, \phi) \in \mathcal{A}^H := \mathcal{A}_E^h \times \Omega^1(\mathfrak{u}E)$ is called a **doubled connection**, where $\mathcal{A}_E^h$ is the space of unitary connections on E with respect to h.*

This is an infinite dimensional affine space modeled on $\Omega^1(\mathfrak{u}E) \oplus \Omega^1(\mathfrak{u}E)$. If $A \in \Omega^0(\mathfrak{u}E)$, then $A^* = -A \in \Omega^0(\mathfrak{u}E)$ and $\star^2 = -1$. This implies that $\star \in \Gamma\mathrm{End}(TX^* \otimes \mathrm{End}E)$ defines a complex structure on $\mathcal{A}_E^h$. If we identify the dual space of $\Omega^1(\mathfrak{u}E)$ with itself via the metric g^h

$$g^h(\dot A, \dot B) = -\int_X \mathrm{tr}(\dot A \wedge \star \dot B) \tag{1}$$

on $\mathcal{A}_E^h$, we obtain a natural complex structure $I_1\colon T_{(D,\phi)}\mathcal{A}^H \to T_{(D,\phi)}\mathcal{A}^H$, $I_1(\dot A, \dot\phi) = (\star\dot A, -\star\dot\phi)$ on $\mathcal{A}^H$. Moreover, $\mathcal{A}^H$ carries the product metric $g := g^h \oplus g^h$ given by

$$g((\dot A_1, \dot\phi_1), (\dot A_2, \dot\phi_2)) := -\int_X \mathrm{tr}(\dot A_1 \wedge \star\dot A_2 + \dot\phi_1 \wedge \star\dot\phi_2)$$

Proposition 2.2. *$(\mathcal{A}^H, I_1)$ and $\mathcal{A}_1^{\mathbb{C}}$ are isomorphic as complex manifolds.*

Proof. Let $f\colon \mathcal{A}^H \to \mathcal{A}_1^{\mathbb{C}}$, $f(D, \phi) = (\bar\partial_D, \phi^{1,0})$, where $\phi^{1,0} = \frac{1}{2}(\phi + i \star \phi)$ and $\bar\partial_D = \frac{1}{2}(D - i \star D)$. f has an explicit inverse by taking the Chern connection and $\varphi \mapsto \varphi - \overline{\varphi}^*$. Let $(\dot A, \dot\phi) \in T_{(D,\phi)}\mathcal{A}^H$, then

$$\begin{aligned} 2df(\star\dot A, -\star\dot\phi) &= (\star\dot A + i\dot A, -\star\dot\phi + i\dot\phi) \\ &= i(\dot A - i\star\dot A, \dot\phi + i\star\dot\phi) \\ &= i \circ 2df(\dot A, \dot\phi). \end{aligned}$$

$\square$

We will use frequently that every element in $\Omega^1(\mathfrak{u}E)$ decomposes uniquely as $\phi = \varphi - \overline{\varphi}^*$ for $\varphi \in \Omega^{1,0}(\mathrm{End}E)$ and vice versa $\varphi = \frac{1}{2}(\phi + i\star\phi)$. The action $\mathcal{G} \circlearrowright \mathcal{A}^H$ is given by conjugation.

Proposition 2.3. *The infinitesimal action of $\mathcal{G}$ at $(D, \phi) \in \mathcal{A}^H$ is given by $\rho\colon \mathfrak{g} \to T_{(D,\phi)}\mathcal{A}^H$, $Z \mapsto (-DZ, [Z, \phi])$.*

Proof. It suffice to consider the matrix exponential up to first order terms under the differentiation. Using that $1 \in \Omega^0(UE)$ is parallel we obtain

$$\begin{aligned} \rho(Z) &= \frac{d}{dt}\Big|_{t=0}(1 + tZ).(D, \phi) \\ &= \frac{d}{dt}\Big|_{t=0}(DtZ + \mathcal{O}(t^2), \phi - \phi tZ + tZ\phi + \mathcal{O}(t^2)) \\ &= (-DZ, [Z, \phi]). \end{aligned}$$

$\square$

I_1 is just one of many complex structures on the space of doubled connections. In fact we have the following statement:

Proposition 2.4. *The tuple (I_1, I_2, I_3, g), with $I_2(\dot A, \dot\phi) = (-\dot\phi, \dot A)$ and $I_3(\dot A, \dot\phi) = (-\star\dot\phi, -\star\dot A)$, defines a hyperkähler structure on $\mathcal{A}^H$ with Kähler forms $\omega_i = g(I_i\,\cdot, \cdot)$. Further the map $\mu = (\mu_1, \mu_2, \mu_3)\colon \mathcal{A}^H \to (\mathfrak{g}^*)^{\oplus 3}$, where*

(i) $\mu_1(D, \phi) := -F^D + \phi \wedge \phi - 2\pi \mathrm{i}\frac{d}{n}\mathrm{id}_E\omega_X$

(ii) $\mu_2(D, \phi) := -D\star\phi$

(iii) $\mu_3(D, \phi) := D\phi$

is a hyperkähler moment map for $\mathcal{G} \circlearrowright \mathcal{A}^H$.

Proof. First note that $\mathcal{A}^H$ is an affine space and all commutators of vector fields vanish, hence the almost complex structures are integrable. Further the $\star^2 = -1$ implies the quaternionic relation $I_i^2 = I_1 I_2 I_3 = -1$. The forms ω_i have constant coefficients and are therefore closed. Let $\{e_k\}$ be a basis of $\Omega^1(\mathfrak{u}E)$. A frame of $T\mathcal{A}^H$ is given by $p = (D, \phi) \mapsto \epsilon_{kl}(p) = (e_k, e_l)$. Thus, the coefficient functions $\omega_{i,klmn} = \omega_i(\epsilon_{kl}, \epsilon_{mn})$ are constant. It remains to check that g is hermitian with respect to all I_i. Let $Z_i = (\dot A_i, \dot\phi_i) \in T_{(D,\phi)}\mathcal{A}^H$ for $i = 1, 2$ then

$$\begin{aligned} g(I_1 Z_1, I_1 Z_2) &= -\int_X \mathrm{tr}(\star\dot A_1 \wedge \star^2 \dot A_2 + (-\star)\dot\phi_1 \wedge \star(-\star)\dot\phi_2) \\ &= -\int_X \mathrm{tr}(\dot A_1 \wedge \star\dot A_2 + \dot\phi_1 \wedge \star\dot\phi_2) \\ &= g(Z_1, Z_2). \end{aligned}$$

Using that $g = g^h \oplus g^h$, where g^h as in 1, we obtain

$$\begin{aligned} g(I_2 Z_1, I_2 Z_2) &= g((-\dot\phi_1, \dot A_1), (-\dot\phi_2, \dot A_2)) \\ &= g^h(-\dot\phi_1, -\dot\phi_2) + g^h(\dot A_1, \dot A_2) \\ &= g((\dot A_1, \dot\phi_1), (\dot A_2, \dot\phi_2)) \\ &= g(Z_1, Z_2). \end{aligned}$$

Combing the calculations above

$$\begin{aligned}g(I_3Z_1, I_3Z_2) &= g((-\star\dot{\phi}_1, -\star\dot{A}_1), (-\star\dot{\phi}_2, -\star\dot{A}_2))\\ &= g((\star\dot{A}_1, -\star\dot{\phi}_1), (\star\dot{A}_2, -\star\dot{\phi}_2))\\ &= g(I_2Z_1, I_2Z_2)\\ &= g(Z_1, Z_2).\end{aligned}$$

Thus, (I_1, I_2, I_3, g) is a hyperkähler structure. Now let $(\dot{A}, \dot{\phi}) \in T_{(D,A)}\mathcal{A}^H$ and $Z \in \mathfrak{g}$. We need to check that $d\mu_i(\dot{A}, \dot{\phi})(Z) = \omega_i(\rho(Z), (\dot{A}, \dot{\phi}))$ for all $i = 1, 2, 3$.

$$\begin{aligned}d\mu_1(\dot{A}, \dot{\phi}) &= \frac{d}{dt}\Big|_{t=0}\mu_1(D + t\dot{A}, \phi + t\dot{\phi})\\ &= \frac{d}{dt}\Big|_{t=0} - F^{D+t\dot{\phi}} + (\phi + t\dot{\phi}) \wedge (\phi + t\dot{\phi})\\ &= \frac{d}{dt}\Big|_{t=0} - F^D - tD\dot{A} + \phi \wedge \phi + t[\dot{\phi}, \phi] + \mathcal{O}(t^2)\\ &= -D\dot{A} + [\dot{\phi}, \phi].\end{aligned}$$

A local computation and the properties of the trace yields $\mathrm{tr}(Z[\dot{\phi}, \phi]) = \mathrm{tr}([Z, \phi] \wedge \dot{\phi})$. Using this equation, Stokes theorem and that the trace is parallel we obtain

$$\begin{aligned}d\mu_1(\dot{A}, \dot{\phi})(Z) &= \int_X \mathrm{tr}(-ZD\dot{A} + Z[\dot{\phi}, \phi])\\ &= \int_X \mathrm{tr}(DZ \wedge \dot{A} + Z[\dot{\phi}, \phi])\\ &= \int_X \mathrm{tr}(DZ \wedge \dot{A} + [Z, \phi] \wedge \dot{\phi})\\ &= \omega_1(\rho(Z), (\dot{A}, \dot{\phi})).\end{aligned}$$

Note that the of $D + t\dot{A}$ induced connection on EndE is of the form $D + t[\dot{A}, \cdot]$. Then

$$\begin{aligned}d\mu_2(\dot{A}, \dot{\phi}) &= \frac{d}{dt}\Big|_{t=0}\mu_2(D + t\dot{A}, \phi + t\dot{\phi})\\ &= \frac{d}{dt}\Big|_{t=0}(-D - t\dot{A})(\star\phi + t\star\dot{\phi})\\ &= \frac{d}{dt}\Big|_{t=0} - D\star\phi - tD\star\dot{\phi} - t[\dot{A}, \star\phi] + \mathcal{O}(t^2)\\ &= -D\star\phi - [\dot{A}, \star\phi].\end{aligned}$$

We conclude

$$\begin{aligned}d\mu_2(\dot{A}, \dot{\phi})(Z) &= \int_X \mathrm{tr}(-ZD\star\dot{\phi} - Z[\dot{A}, \star\phi])\\ &= \int_X \mathrm{tr}(DZ \wedge \star\dot{\phi} - \star[Z, \phi] \wedge \dot{A})\\ &= \int_X \mathrm{tr}(DZ \wedge \star\dot{\phi} + [Z, \phi] \wedge \star\dot{A})\\ &= \omega_2(\rho(Z), (\dot{A}, \dot{\phi})).\end{aligned}$$

$i = 3$ can be shown with a similar calculation. The adjoint action of $\mathcal{G}$ on its Lie algebra is given by conjugation and with the trace being invariant under permutation we obtain for each $i = 1, 2, 3$

$$\begin{aligned}Ad_g^*\mu_i(D, \phi)(Z) &= \mu_i(D, \phi)(g^{-1}Zg)\\ &= \mu_i(gDg^{-1}, g\phi g^{-1}).\end{aligned}$$

Thus, the maps are $\mathcal{G}-$equivariant.

□

We call a doubled connection **harmonic** if it is contained in $\mu^{-1}(0)$ and **irreducible** if there are no proper D- and ϕ-invariant subbundles of E. Denote the set of irreducible double connections as $\mathcal{A}^{H,s}$.

Theorem 2.5. *The action $\mathcal{G}_{\mathrm{eff}} := \mathcal{G}/U(1) \circlearrowright \mathcal{A}^{H,s}$ is free and proper and for all harmonic $(D, \phi) \in \mathcal{A}^{H,s}$ we have that*

$$T_{(D,\phi)}(\mathcal{G}.(D, \phi)) \oplus (T_{(D,\phi)}(\mathcal{G}.(D, \phi)))^\perp = T_{(D,\phi)}\mu^{-1}(0).$$

Proof. In [AN16, Ch. 6.3]. □

Set $\mu_s := \mu|_{\mathcal{A}^{H,s}}$, then the hyperkähler quotient $\mathcal{M}^s_{n,d}(X) := \mu_s^{-1}(0)/\mathcal{G}_{\mathrm{eff}}$ is called the **moduli space of irreducible harmonic doubled connections** and is a smooth manifold inheriting the hyperkähler structure of $\mathcal{A}^H$. Without the irreducibilty we get the **moduli space of harmonic double connections** $\mathcal{M}_{n,d}(X)$, which is a priori not a smooth manifold. The tangent spaces to $\mathcal{M}^s_{n,d}(X)$ are given by

$$\begin{aligned}T_{[(D,\phi)]}\mathcal{M}^s_{n,d}(X) &= \ker(d_{(D,\phi)}\mu_s)/\rho(\mathfrak{g})\\ &\cong \ker(d_{(D,\phi)}\mu_s) \cap \rho(\mathfrak{g})^\perp.\end{aligned}$$

Let us calculate the orhorgonal complement of $\rho(\mathfrak{g}) \subset T_{(D,\phi)}\mathcal{A}^H$ with respect to g. Let $Z \in \mathfrak{g}$ and $(\dot{A}, \dot{\phi}) \in T_{(D,\phi)}A^H$. Then

$$\begin{aligned}g(\rho(Z), (\dot{A}, \dot{\phi})) &= -\int_X \mathrm{tr}(-DZ \wedge \star\dot{A} + [Z, \phi] \wedge \star\dot{\phi})\\ &= \int_X \mathrm{tr}(Z(D\star\dot{A} + [\phi, \star\dot{\phi}]).\end{aligned}$$

Therefore, $(\dot{A}, \dot{\phi}) \in \rho(\mathfrak{g})^\perp$ if and only if $D\star\dot{A} = -[\phi, \star\dot{\phi}]$.

Proposition 2.6. *The quaternonic dimension of the hyperkähler manifold $\mathcal{M}^s_{n,d}(X)$ is $(g-1)n^2 + 1$.*

Proof. We won't give the details but only an idea of the proof. For details check [AN16, Ch. 6.3]. Consider the map $\widehat{D}\colon T_{(D,\phi)}\mathcal{A}^H \to \Omega^2(\mathfrak{u}E)^{\oplus 4}$ given by

$$\widehat{D}(\dot{A}, \dot{\phi}) = (d_{(D,\phi)}\mu_s(\dot{A}, \dot{\phi}), D\star\dot{A} + [\phi, \star\dot{\phi}]).$$

One can show that this is an elliptic operator of index $(g-1)n^2$ and its cokernel has quaternionic dimension 1. Then, $\dim_{\mathbb{H}} \ker \widehat{D} = \mathrm{ind}\widehat{D} + \dim_{\mathbb{H}} \mathrm{coker}\widehat{D} = (g-1)n^2 + 1$. The proposition follows from the fact that the tangent space to the moduli space at a point (D, ϕ) is exactly the kernel of $\widehat{D}$. □

3. THE MODULI SPACE OF HIGGS BUNDLES

Now we want to specify the zero set of μ in a reasonable way to define the moduli of Higgs bundles, i.e. consider it as a subset of $\mathcal{A}_1^{\mathbb{C}}$. Let us combine two of the moment maps as $\mu_{\mathbb{C}} = \mu_3 + i\mu_2 \colon \mathcal{A}^H \to \mathfrak{g}_{\mathbb{C}}^*$. Using that the holomorphic structure of a connection is given by $\bar{\partial}_D = \frac{1}{2}(D - i \star D)$ and $\phi = \varphi - \overline{\varphi}^*$ we obtain

$$\tilde{\mu}(D,\phi) = (D - i \star D)(\varphi - \overline{\varphi}^*) = 2\bar{\partial}_D \varphi.$$

Further $\phi \wedge \phi = -\varphi \wedge \overline{\varphi}^* - \overline{\varphi}^* \wedge \varphi = -[\varphi, \overline{\varphi}^*]$ and therefore

$$\mu^{-1}(0) = \{(D,\phi) | \bar{\partial}_D \varphi = 0,\, F^D + [\varphi, \overline{\varphi}^*] = -2\pi i \frac{d}{n} \mathrm{id}_E \omega_X\}.$$

Thus, under the isomorphism f from proposition 2.2 the zero set of μ are Higgs bundles $(\bar{\partial}, \varphi) \in \mathcal{A}_1^{\mathbb{C}}$ such that the Chern connection can be modified to a projectivly flat connection $\nabla = D^{\bar{\partial}} + \varphi + \overline{\varphi}^*$, i.e.

$$F^{\nabla} = -2\pi i \frac{d}{n} \mathrm{id}_E \omega_X.$$

We have the following theorem, proven by Carlos T. Simpson in [CS88], connecting the notion of stability of Higgs bundles and irreducibility of doubled connections.

Theorem 3.1. *For all $[(D,\phi)] \in \mathcal{M}_{n,d}^s(X)$ the pair $(\bar{\partial}_D, \varphi)$ is a stable Higgs bundle and vice versa every stable Higgs bundle is isomorphic to $(\bar{\partial}_D, \varphi)$ for a irreducible doubled connection (D,ϕ). Moreover, polystable Higgs bundles correspond in the same way to harmonic doubled connections.*

Thus, we have a one-to-one correspondence of isomorphism classes of stable Higgs bundles and points in $\mathcal{M}_{n,d}^s(X)$. Finally we are ready to state the following definition.

Definition 3.2. *The **moduli space of stable Higgs bundles** of rank n and degree d over a compact Riemann surface X of genus $g \geq 2$ is the smooth Kähler manifold of complex dimension $2(g-1)n^2 + 2$ given by*

$$\mathcal{H}_{n,d}^s(X) := (\mathcal{M}_{n,d}^s(X), I_1, g).$$

Without the irreduciblity we have the **moduli of polystable Higgs bundles** $\mathcal{H}_{n,d}(X)$ which is a quasi-projective variety with singularities and its smooth points are exactly the isomorphism classes of stable Higgs bundles.

Example 3.1. ***Higgs line bundles of degree 0***
First realize that a holomorphic line bundle has no proper holomorphic subbundles and hence every Higgs bundle of rank 1 is stable. Consider a line bundle $L \to X$ of degree zero together with a Higgs bundle $(\bar{\partial}, \varphi)$. Since, $\mathrm{EndL} = L^{-1}L = \underline{\mathbb{C}}_X$ *a Higgs field is a holomorphic section of the canonical bundle $\varphi \in H^0 K$. Under the isomorphism*

$$\Omega^1(X) \ni \phi \mapsto \phi + i \star \phi \in \Gamma K$$

we have $H^0 K \cong \mathrm{Harm}(X) = \ker d \cap \ker \star d$ and via the Hodge-decomposition the space of harmonic 1-forms is isomorphic to $H^1_{dR}(X)$, which is a $2g$-dimensional real vector space. Let (D,ϕ) be a harmonic doubled connection corresponding to the Higgs bundle. Then

$$\begin{aligned}\mu^{-1}(0) &= \{(D,\phi) \in \mathcal{A}^H \mid F^D = 0, \phi \in \mathrm{Harm}X\} \\ &= \widetilde{\mu}^{-1}(0) \times \mathrm{Harm}X,\end{aligned}$$

where $\widetilde{\mu} \colon \mathcal{A}^h \to \mathfrak{g}^*$, $D \mapsto -F^D$ is the moment map of the action $\mathcal{G} \circlearrowright \mathcal{A}^h$. The Kähler quotient $(\widetilde{\mu}^{-1}(0)/\mathcal{G}, \star)$ is the moduli space of holomorphic structures [AN16, Chapter 5]. Since $\mathcal{G}$ acts trivial on 1-forms we obtain

$$\begin{aligned}\mathcal{H}_{1,0}(X) &= (\widetilde{\mu}^{-1}(0)/\mathcal{G}, \star) \times (\mathrm{Harm}(X), -\star) \\ &= \mathrm{Jac}X \times (\mathbb{C}^g)^* \\ &= T^*\mathrm{Jac}X.\end{aligned}$$

Thus, the moduli space of degree zero and rank one Higgs bundles over a compact Riemann surface is a smooth Kähler manifold given by the cotangent bundle to the Jacobian variety of X.

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