Harmonic bundles

Preliminary considerations: Let M be complex manifold, let $E \to M$ be a complex VB. Then a hol. structure on E is operator $\bar\partial_E : E \to E \otimes A^{0,1}_M$ with $\bar\partial_E^2 = 0$, namely $\mathcal{E} := \ker(\bar\partial)$ is hol. VB on M.

If $h : E \otimes \bar{E} \to C^\infty_M$ is hermitian, then $\forall\, \bar\partial_E$ hol. str. $\exists$ unique $\partial_E : E \to E \otimes A^{1,0}_M$ s.t.: 1.) $\bar\partial_E^2 = 0$, 2.) $d_E := \partial_E + \bar\partial_E$ is h-metric (or unitary), i.e. $d\,h(s,t) = h(d_E s, t) + h(s, d_E t)$ for $s,t \in E$ (i.e. $s,t \in \Gamma(M,E)$), d_E is called Chern connection of $(h, \bar\partial_E)$. However,

$$d_E^2 \neq 0, \text{ but } d_E^2 = (\partial_E + \bar\partial_E)\circ(\partial_E + \bar\partial_E) =$$

$$\partial_E^2 + [\partial_E, \bar\partial_E] + \bar\partial_E^2 = [\partial_E, \bar\partial_E]$$

Since d_E is not flat, this is not related to representation of $\pi_1(X)$. How to circumvent this?

Let now $\mathcal{E}$ be hol. bundle on M (i.e. $\mathcal{E} = \ker(\bar\partial_E)$ for some $\bar\partial_E : E \to E \otimes A^{0,1}_M$).

Then $\theta \in \Gamma(M, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_M)$ with $\theta^2 = \theta \wedge \theta = 0$ is called Higgs bundle, i.e. $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_M$ which is linear over $\mathcal{O}_M$.

Fact (easy): Let $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_M$ be any hol. connection (flat or not), then $\nabla + \theta$ is also connection (check Leibniz rule).

rk: (connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_M$) $\rightsquigarrow$ ($\partial_E : E \to E \otimes A^{1,0}_M$ if $\mathcal{E} = \ker(\bar\partial_E)$)

$$\nabla \text{ flat, i.e. } \nabla^2 = 0 \iff \partial_E^2 = 0 \text{ and } [\partial_E, \bar\partial_E] = 0$$

notice: $\dim M = 1 \Rightarrow A^{k,0}_M = A^{0,k}_M = 0 \quad \forall k > 1$

hence $\partial_E^2 = \bar\partial_E^2 = 0$ is automatic

<u>Def.</u>: Let M be a complex manifold, and let $E \to M$ be a complex vector bundle.
Let $h: E \otimes \overline{E} \to \mathcal{C}^\infty_M$ be hermitean + pos. def.. Then h is called a harmonic metric (and (E,h) a harmonic bundle) iff $\exists$ operators

$$\partial, \theta : E \longrightarrow E \otimes \mathcal{A}^{1,0}_M \text{ and } \bar\partial, \bar\theta : E \longrightarrow E \otimes \mathcal{A}^{0,1}_M$$

s.t.: 1.) $\theta, \bar\theta$ are $\mathcal{C}^\infty_M$-linear, $\partial, \bar\partial$ satisfy Leibniz

2.) $(\partial + \theta + \bar\partial + \bar\theta)^2 = 0$ (equation in $E \otimes \mathcal{A}^2_M$)

3.) $\partial + \bar\partial$ is h-unitary, $\theta + \bar\theta$ is h-self-adjoint.

4.) $\bar\partial(\theta) := [\bar\partial, \theta] = 0$ $\left(\Leftrightarrow [\partial, \bar\theta] = 0\right)$

---

with some work: 2.) + 3.) $\Rightarrow$

$$\partial^2 = \theta^2 = \bar\theta^2 = \bar\partial^2 = 0 \quad (*) \quad \text{and} \quad [\partial, \theta] = [\bar\partial, \bar\theta] = 0$$

notice: $\dim_{\mathbb{C}}(M) = 1 \Rightarrow (*)$ is automatic

(4) $\Longrightarrow$ θ induces on $\mathcal{E} := \ker(\bar\partial)$ a Higgs field

$$\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_M$$

From now on: $M = \Sigma$ compact Riemann surface $g > 1$, $\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ universal cover ($\widetilde{\Sigma}$ unit disc).

<u>aim</u>: given flat bundle $(E, D = D' + D'')$ on Σ, construct hermitian metric h s.t. $D' = \partial + \theta$, $D'' = \bar{\partial} + \bar{\theta}$ as above, i.e. $(E, h, \bar{\partial}, \theta)$ is harmonic.

<u>Idea</u>: $\pi^* E$ is flat, let $V := \Gamma(\widetilde{\Sigma}, \ker \pi^* D)$ be space of multivalued flat sections

Fix identification $V \simeq \mathbb{C}^n$, then pos. def. metric h on $\pi^* E$ is a map $u_h : \widetilde{\Sigma} \longrightarrow \mathrm{PHerm}(n)$

<u>notice</u>: $\mathrm{PHerm}(n) := \{ H \in gl_n \mid H = \bar{H}^{tr}, \text{ pos. def.} \}$ has transitive $Gl_n(\mathbb{C})$-action via $g.H = \bar{g}^{tr} \cdot H \cdot g$ with $\mathrm{Stab}(id) = U(n) \rightsquigarrow \mathrm{PHerm} \simeq Gl_n(\mathbb{C}) / U(n)$

<u>notice</u>: $\pi_1(\Sigma)$ acts on $\widetilde{\Sigma}$ via deck transformation and on $\mathrm{PHerm}(n)$ via $\rho \in \mathrm{Rep}(\pi_1(\Sigma), Gl_n(\mathbb{C}))$ corresponding to (E, D)

<u>Claim</u>: pos. def. hermitean metric on E $\Longleftrightarrow$

ρ-equivariant C^∞-map $u_h : \tilde{\Sigma} \to Gl(n)/U(n)$, i.e.

$$u_h(x\gamma) = u_h(x)\,\rho(\gamma) \quad \forall \gamma \in \pi_1(\Sigma)$$

<u>Theorem (Corlette-Donaldson)</u>: (E, D) is semi-simple, i.e. $\rho : \pi_1(\Sigma) \to Gl_n(\mathbb{C})$ is semi-simple $\Longleftrightarrow$

$\exists$ harmonic metric on E (unique up to scaling)

<u>Tool</u>: harmonic maps: Let (M,g), (N,g') compact Riemannian.

$f \in C^\infty(M,N)$, then $df \in \Gamma(M, T^*M \otimes f^*TN)$

from g, g' $\rightsquigarrow$ metric $|\cdot|_{(g,g')}$ on $T^*M \otimes f^*TN$

Put $E(f) := \int_M |df|^2_{(g,g')} \operatorname{vol}(M)$ "energy of f"

Then f harmonic $:\underset{\text{Df.}}{\Longleftrightarrow}$ "f is critical pt. for $E(f)$"

$$\Longleftrightarrow \forall v \in f^*TN : D_v E(f) := \frac{d E(f_t)}{dt}\Big|_{t=0} = 0 \,,$$

where $f_t : M \to N$ s.t. $f_0 = f$ & $\frac{df_t}{dt}\Big|_{t=0} = v$

<u>Theorem (Eells-Sampson)</u>: If (N,g) has sectional curvature ≤ 0, then $\forall\, [M,N]$ (homotopy class), $\exists!\; f: M \to N$ $[f] \in [M,N]$ s.t. f is harmonic.

<u>remark</u>: in our situation $u_h : \tilde{\Sigma} \to Gl(n)/U(n)$, source + target mf. are not compact.

however, theory works for equivariant maps

<u>notice also</u>: from theory of homogeneous spaces: G/K with K max. compact is of "non-compact type"

$\Rightarrow$ sectional curvature ≤ 0

<u>Claim</u> (not to hard): (E,D) flat, then h is harmonic

$\Leftrightarrow$ $u_h : \tilde{\Sigma} \to Gl(n)/U(n)$, $p \mapsto h(e_i, e_j)$ is harmonic map

How does semi-simplicity comes into play?

Let (E,D) be flat, $F \subsetneq E$ flat subbundle.

Let h be hermitean $\Rightarrow E = F \oplus F^\perp$

(as $\mathcal{C}^\infty$-bundles). We have $D = \underset{\substack{\uparrow \\ \text{unitary}}}{d} + \underset{\substack{\uparrow \\ \text{self-adjoint}}}{\Theta}$ and

$$D = \begin{pmatrix} D_1 & \eta \\ 0 & D_2 \end{pmatrix} \text{ where } \eta : F^\perp \to F \otimes \Omega^1$$

Notice: η is $\mathcal{C}^\infty$-linear, since $\forall x \in F^\perp, f \in \mathcal{C}^\infty$

$$D(f \cdot x) = f \cdot Dx + \underbrace{df \cdot x}_{\in F^\perp} \Rightarrow \eta(f \cdot x) = f \cdot \eta(x)$$

(E,D) is semi-simple if $\eta = 0$ $(\forall F)$

decompose further $D_i = d_i + \Theta_i$, then

$$D = \begin{pmatrix} D_1 & \eta \\ 0 & D_2 \end{pmatrix} = \underbrace{\begin{pmatrix} d_1 & \eta/2 \\ -\eta^*/2 & d_2 \end{pmatrix}}_{\text{unitary}} + \underbrace{\begin{pmatrix} \Theta_1 & \eta/2 \\ \eta^*/2 & \Theta_2 \end{pmatrix}}_{\text{self-adjoint}}$$

Consider bundle $\mathrm{End}(E)$ and section

$$s = \begin{pmatrix} -id_F & 0 \\ 0 & id_{F^\perp} \end{pmatrix} \in \mathrm{End}(E).$$

induced connection $D^{\mathrm{End}(E)} = d^{\mathrm{End}(E)} + \Theta^{\mathrm{End}(E)}$

$$d^{\mathrm{End}(E)}(s) = \underbrace{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} -id_F & 0 \\ 0 & id_{F^\perp} \end{pmatrix}}_{\substack{=0 \text{ since} \\ id_F \text{ and } id_{F^\perp} \text{ flat}}} + \underbrace{\left[\begin{pmatrix} 0 & \eta/2 \\ -\eta^*/2 & 0 \end{pmatrix}, \begin{pmatrix} -id_F & 0 \\ 0 & id_{F^\perp} \end{pmatrix}\right]}_{= \begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix}}$$

<u>Then:</u> $\langle \Theta, d^{\mathrm{End}(E)} s \rangle_{L^2} :=$

Θ: self-adjoint part of connection D on $\mathrm{End}(E)$

$$\int_\Sigma \langle \underset{\mathcal{A}^1 \otimes \mathrm{End}(E)}{\Theta}, d^{\mathrm{End}(E)} s \rangle_h = \int_\Sigma \mathrm{tr}\begin{pmatrix} \Theta_1 & \eta/2 \\ -\eta^*/2 & \Theta_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix}$$

$$= \int_\Sigma \eta \cdot \eta^* = \langle \eta, \eta \rangle_{L^2}$$

hence $\eta = 0 \iff \langle \Theta, d^{End(E)} s \rangle_{L^2} = \langle d^{End(E),*} \Theta, s \rangle_{L^2} = 0$

where $d^{End(E),*}$ is h-adjoint of $d^{End(E)}$

so: $d^{End(E),*} \Theta = 0 \implies \eta = 0$

Θ self-adj.

ex./problem: $d^{End(E),*} \Theta = 0 \iff d^{End(E)} \Theta = 0$

<u>hence</u>: $(E, h, D = d + \Theta)$ harmonic

(i.e. $d\Theta = 0$, i.e. $\bar{\partial}\theta = 0$ if

$d = \partial + \bar{\partial}$ and $\Theta = \theta + \bar{\theta}$) $\implies$

(E, D) is semi-simple