The Hitchin-Simpson Correspondence

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Abstract

In this talk, we give a proof sketch of the Hitchin-Simpson correspondence following Wentworth [Wen16], which states that to a fixed complex vector bundle $E \to X$ over a compact Riemann surface $g \geq 2$ up to smooth bundle isomorphism, the moduli space of polystable Higgs bundles $(\overline{\partial}_E, \Phi)$ is homeomorphic to the moduli space of harmonic metrics on $(\overline{\partial}_E, \Phi)$, that is, hermitian metrics h, whose associated Chern connection paired with the Higgs field (A, Φ) satisfies Hitchins equation

$$\star(F_A + [\Phi, \Phi^*]) = -i\mu(E)\,\mathrm{Id}_E.$$

We begin by fixing a complex vector bundle $E \to X$ over a compact Riemann surface $g \geq 2$ with rank n and degree d.

Remark. Any complex vector bundle over a compact Riemann surface with rank n and degree d is smoothly isomorphic to

$$E \cong \det(E) \oplus \mathbb{C}^{n-1},$$

where $\det(E)$ is the determinant line bundle of E, which has degree d. Furthermore, complex line bundles are classified by their first Chern class, or equivalently, when the base is a compact curve, by their degree. Hence fixing the rank and degree of E completely determines its smooth type.

Structures on E

Recall the definition of the following structures on $E \to X$:

Definition. A **Hermitian metric** h on E is an assignment of a Hermitian inner product

$$h_p : E_p \times E_p \to \mathbb{C}$$

that varies smoothly along the base X.

Remark.

1. The existence of a Hermitian metric on E is equivalent to a reduction of structure group from $\mathrm{GL}(n, \mathbb{C})$ to $U(n)$. Such a reduction is always possible as $U(n)$ is the maximal compact subgroup of $\mathrm{GL}(n, \mathbb{C})$, and $\mathrm{GL}(n, \mathbb{C})$ deformation retracts onto $U(n)$.
2. The **unitary gauge group** $\mathcal{G}_{(E,h)} := \{g \in \mathcal{A}^0(X, \mathrm{Aut}(E)), g^* = g^{-1})\}$ is the group of automorphisms of E preserving its Hermitian structure

$$h_p(g_p \cdot u, g_p \cdot v) = h_p(u, v), \quad u, v \in E_p.$$

For Hermitian metrics h, h', there is a natural isomorphism $\mathcal{G}_{(E,h)} \cong \mathcal{G}_{(E,h')}$. We will henceforth drop the dependence on h and refer to the unitary gauge group as $\mathcal{G}_E$.

Definition. A **holomorphic structure** on E is a Dolbeault operator

$$\overline{\partial}_E : \mathcal{A}^0(X, E) \to \mathcal{A}^{0,1}(X, E),$$

such that $\overline{\partial}_E^2 = 0$ and satisfies the Leibniz rule

$$\overline{\partial}_E(f \otimes s) = \overline{\partial} f \otimes s + f \otimes \overline{\partial}_E s,$$

where $\overline{\partial} : \mathcal{A}^0(X) \to \mathcal{A}^{0,1}(X)$ is the holomorphic structure of the base X.

Remark.

1. A holomorphic structure $\overline{\partial}_E$ is a complex structure on the total space E, which makes it into a complex manifold. We denote the space of holomorphic structures on E by $\mathrm{Dol}(E)$.

2. The **complex gauge group** $\mathcal{G}_E^{\mathbb{C}} = \{g \in \mathcal{A}^0(X, \mathrm{Aut}(E))\}$ is the group of automorphisms of E acting on the holomorphic structures via

$$g \cdot (E, \overline{\partial}_E) = (E, g \circ \overline{\partial}_E \circ g^{-1}).$$

   Two holomorphic vector bundles are holomorphically isomorphic if and only if they are related by a complex gauge group action.

3. Let $(E, \overline{\partial}_E, h)$ be a **holomorphic Hermitian vector bundle**. Since $\mathcal{G}_E \subseteq \mathcal{G}_E^{\mathbb{C}}$, the holomorphic and Hermitian structure stays in the same isomorphism class within the unitary gauge orbit. However, within the complex gauge orbit, the Hermitian structure varies. In fact, we can identify the space of Hermitian structures with $\mathcal{G}_E^{\mathbb{C}}/\mathcal{G}_E$.

Definition. A **connection** on E is a differential operator

$$d_A : \mathcal{A}^0(X, E) \to \mathcal{A}^1(X, E)$$

satisfying the Leibniz rule

$$d_A(f \otimes s) = df \otimes s + f \otimes d_A s,$$

where $d : \mathcal{A}^0(X) \to \mathcal{A}^1(X)$ is exterior differentiation on X.

Remark.

1. A connection is called **flat** if its curvature vanishes $F_A = d_A^2 = 0 \in \mathcal{A}^2(X, \mathrm{End}(E))$.

2. The unitary and complex gauge group acts on d_A via conjugation, $g \cdot d_A = g \circ d_A \circ g^{-1}$.

3. Given a Hermitian structure (E, h), d_A is called a **unitary connection** if $d_A h = 0$, i.e. h is parallel with respect to d_A. We will denote the space of unitary connections with respect to h by $\mathcal{A}(E, h)$.

The Atiyah-Bott Isomorphism

The three structures above are a priori independent of each other, each one of them can exist on the smooth complex bundle E without any relation to the other two. However, the Atiyah-Bott isomorphism characterizes a relationship between holomorphic structures and unitary connections on E.

Theorem (Atiyah-Bott Isomorphism)**.** *Fix (E, h) a Hermitian structure. Then there is a one to one correspondence:*

$$\mathrm{Dol}(E) \overset{1:1}{\longleftrightarrow} \mathcal{A}(E, h)$$

between the set of holomorphic structures $\overline{\partial}_E$ on E and the set of unitary connections with respect to h.

Proof. (⇒) Given a holomorphic structure $\overline{\partial}_E$, there is a unique unitary connection d_A such that $d_A^{0,1} = \overline{\partial}_E$ called the **Chern connection**.

(⇐) Conversely, given a unitary connection d_A, its $(0,1)$-part $d_A^{0,1}$ satisfies the Leibniz rule and $(d^{0,1})^2 = 0$ as there are no nontrivial $(0,2)$-forms on compact Riemann surfaces.

□

Note that given a holomorphic structure $(E, \overline{\partial}_E)$, the choice of a Hermitian metric h determines uniquely the Chern connection $A(\overline{\partial}_E, h) =: A$.

Definition. A Hermitian metric h on $(E, \overline{\partial}_E)$ is called a **Hermite-Einstein metric** if its associated Chern connection A is projectively flat,

$$\star F_A = -i\mu(E)\,\mathrm{Id}_E,$$

where $\mu(E) := \deg(E)/\mathrm{rank}(E)$ is the slope of E and $\mathrm{Id}_E \in \mathcal{A}^0(X, \mathrm{End}(E))$ is section of identity matrices at each $p \in X$.

Meanwhile, given a holomorphic vector bundle $(E, \overline{\partial}_E)$, a Higgs field is a holomorphic section $\Phi \in H^0(X, \mathrm{End(E)} \otimes K)$, where $K := T^*X$ denotes the canonical line bundle. Together, a pair $(E, \overline{\partial}_E, \Phi)$ is called a **Higgs bundle**.

Definition. A Hermitian metric h on $(E, \overline{\partial}_E, \Phi)$ is called a **harmonic metric** if its associated Chern connection and Higgs field pair (A, Φ) satisfies the **Hitchin equation**,

$$\star(F_A + [\Phi, \Phi^*]) = -i\mu(E)\,\mathrm{Id}_E.$$

The Hitchin-Simpson Correspondence

The Hitchin-Simpson correspondence relates the polystability of a Higgs bundle to the existence of a harmonic metric on it, while the Hitchin-Kobayashi correspondence is a simpler version relating the polystability of a holomorphic vector bundle to the existence of a Hermite-Einstein metric on it. Their proofs are similar in vein, where the former essentially builds on that of the latter, but accounting for the extra Higgs field. In the case of vanishing Higgs field $\Phi \equiv 0$, the Hitchin-Simpson correspondence reduces to the Hitchin-Kobayashi correspondence.

Before stating the correspondence, let's briefly recall the notion of stability for vector bundles and Higgs bundles. A holomorphic vector bundle is:

- **stable** if all holomorphic subbundles F has strictly smaller slope $\mu(F) < \mu(E)$,
- **polystable** if it is the direct sum of stable subbundles $E = \bigoplus_i E_i$ of the same slope $\mu(E) = \mu(E_i)$.

On the other hand, the stability conditions for Higgs bundles are restricted to Φ-invariant subbundles $F \subset E$, $\Phi(F) \subset F \otimes K$.

A Higgs bundle is:

- **stable** if all Φ-invariant subbundles F has strictly smaller slope $\mu(F) < \mu(E)$,

- **polystable** if it is the direct sum of stable Higgs subbundles $(E, \Phi) = \bigoplus_i (E_i, \Phi_i)$ of the same slope $\mu(E) = \mu(E_i)$.

Theorem (Hitchin-Kobayashi Correspondence)**.** *Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle, then*

$$(E, \overline{\partial}_E) \text{ polystable } \Leftrightarrow (E, \overline{\partial}_E) \text{ admits a Hermite-Einstein metric.}$$

For degree zero, $(E, \overline{\partial}_E)$ is polystable if and only if E admits a flat connection.

Theorem (Hitchin-Simpson Correspondence)**.** *Let $(E, \overline{\partial}_E, \Phi)$ be a Higgs bundle, then*

$$(E, \overline{\partial}_E, \Phi) \text{ polystable } \Leftrightarrow (E, \overline{\partial}_E, \Phi) \text{ admits a harmonic metric.}$$

Proof sketch. ($\Leftarrow$) Suppose we have (A, Φ) satisfying Hitchin's equation. Let $S \subset E$ be a Φ-invariant holomorphic subbundle. We show that

$$\mu(S) \leq \mu(E)$$

and equality holds if and only if $\beta = 0$ and Φ splits. This is equivalent to showing that E is polystable.

Using the Hermite-Einstein metric h, we form the decomposition $E = S \oplus Q$ where $Q = E/S$ is a holomorphic bundle that is a subbundle of E precisely when the Dolbeault operator is diagonal,

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_S & \beta \\ 0 & \overline{\partial}_Q \end{pmatrix}.$$

Here $\beta \in H^{0,1}(X, \mathrm{Hom}(Q, S))$ is an extension class for the short exact sequence

$$0 \to S \to E \to Q \to 0,$$

which vanishes $\beta = 0$ when the sequence splits, i.e. when $Q \subset E$ a subbundle.

The associated Chern connection is a block skew-Hermitian form

$$d_A = \overline{\partial}_E - \overline{\partial}_E^* = \begin{pmatrix} d_{A_S} & \beta \\ -\beta^* & d_{A_Q} \end{pmatrix}.$$

We compute its curvature

$$F_A = d_A \wedge d_A = \begin{pmatrix} F_{A_S} - \beta \wedge \beta^* & d_{A_S}\beta + \beta d_{A_Q} \\ -d_{A_Q}\beta^* - \beta^* d_{A_S} & F_{A_Q} - \beta^* \wedge \beta \end{pmatrix}.$$

Recall that the **degree** of a complex vector bundle E over a curve X is the evaluation of its first Chern class

$$c_1(E) = \left[\frac{i}{2\pi} \mathrm{tr} \star F_A\right] \in H^2(X, \mathbb{R})$$

on the volume form ω of X, normalized to $\int_X \omega = 2\pi$.

Let $\pi_S : E \to E$ denote the orthogonal projection onto the subbundle S, $\pi_S^2 = \pi_S$. Recall that on $\mathcal{A}^1(X, \mathrm{End}(E))$ there is a positive definite inner product

$$\langle \alpha, \beta \rangle = -\frac{i}{2\pi} \int_X \mathrm{tr}(\alpha \wedge \star\overline{\beta}).$$

Then, we can compute

$$
\begin{aligned}
\deg S &= \int_X c_1(S)\omega = \frac{i}{2\pi}\int_X \operatorname{tr}(\star F_A)\omega \\
&= \frac{i}{2\pi}\int_X \operatorname{tr}\left(\pi_S \star F_{A_S}\pi_S + \star(\beta\wedge\beta^*)\right)\omega \\
&= \frac{i}{2\pi}\int_X \operatorname{tr}\left(-i\mu(E)\,\pi_S \mathrm{Id}_E \pi_S + i\pi_S \star [\Phi,\Phi^*]\pi_S + \star(\beta\wedge\beta^*)\right)\omega \\
&= \mu(E)\operatorname{rank}(S) - \left(|\pi_S\Phi(I-\pi_S)|^2 + |\beta|^2\right) \leq \mu(E)\operatorname{rank}(S),
\end{aligned}
$$

where in the third line we've used the fact that F_A is a solution to Hitchin's equation, and in the fourth line we've used the computation

$$\frac{i}{2\pi}\int_X \operatorname{tr}(\pi_S \star [\Phi,\Phi^*]\pi_S)\omega = |\pi_S\Phi(I-\pi_S)|^2$$

and that

$$\frac{1}{2\pi}\int_X \operatorname{tr}\star(\beta\wedge\beta^*)\omega = i|\beta|^2.$$

Thus we have shown that

$$\mu(S) \leq \mu(E)$$

with equality if and only if $\beta = 0$ and Φ splits.

($\Rightarrow$) It suffices to assume $\mathcal{E} = (E, \overline{\partial}_E, \Phi)$ is stable. Let A_h be the Chern connection associated to a Hermitian metric h. Then we vary (A_h, Φ) along a sequence in the complex gauge orbit of $\mathcal{E}_\infty = (E, \overline{\partial}_E, \Phi)$ and show that in the limit, there exists a solution (A_∞, Φ_∞) to Hitchin's equation corresponding to $(E_\infty, \overline{\partial}_\infty, \Phi_\infty)$ in the same orbit. We have to show that the limiting bundle $\mathcal{E}_\infty$ is indeed a Higgs bundle and that $\mathcal{E} = \mathcal{E}_\infty$ are isomorphic as Higgs bundles. In other words, the limiting map $g_\infty : \mathcal{E} \to \mathcal{E}_\infty$ of the sequence of gauge transformations $\{g_j\}$ is well-defined gauge transformation.

The proof comes in four steps and involve quite some analysis that we shall gloss over as italicized *claims without proof*. Notably, the proof utilizes Uhlenbeck's weak compactness theorem, which we state here as a proposition directly taken from Wentworth's notes [Wen16]:

Proposition. *Fix $p \geq 2$. Let $\{A_j\}$ be a sequence of L^p_1-connections with $||F_{A_j}||_{L_p}$ uniformly bounded. Then there exists a sequence of unitary gauge transformations $g_j \in L^p_2$ and a smooth unitary connection A_∞ such that (after passing to a subsequence) $g_j(A_j) \to A_\infty$ weakly in L^p_1 and strongly in L^p.*

Step 1. Finding the limiting bundle $(E_\infty, \overline{\partial}_\infty, \Phi_\infty)$: *There exists a minimizing sequence (A_j, Φ_j) for the Yang-Mills-Higgs functional*

$$\mathrm{YMH}(A,\Phi) : \int_X |i\star(F_A + [\Phi,\Phi^*]) - \mu(E)\,\mathrm{Id}_E|^2\omega$$

in the same complex gauge orbit of (E, Φ), for which $||F_{A_j}||_{L^p}$ is bounded for any p. Applying Uhlenbeck's weak compactification we obtain a smooth unitary connection A_∞ such that if we write $\overline{\partial}_{A_j} = \overline{\partial}_{A_\infty} + a_j$, $a_j \to 0$ weakly in L^p_1. By the Sobolev embedding theorem, we may assume that $a_j \to 0$ for some C^α. Each Φ_j is holomorphic by assumption, hence

$$0 = \overline{\partial}_{A_j}\Phi_j = \overline{\partial}_{A_\infty}\Phi_j + [a_j, \Phi_j].$$

Next, using much mathematical analysis, *one argues that Φ_j converges in C^α to some Φ_∞*. By the holomorphicity of Φ_j, we write

$$\overline{\partial}_{A_\infty}\Phi_\infty = \overline{\partial}_{A_j}(\Phi_\infty - \Phi_j) - [a_j, \Phi_j].$$

Since $\Phi_\infty - \Phi_j \to 0$, $[a_j, \Phi_j] \to 0$ in C^α, we have that $\overline{\partial}_{A_\infty}\Phi_\infty = 0$ weakly. Hence, by **Weyl's lemma**, Φ_∞ is holomorphic, and $(E_\infty, \overline{\partial}_\infty, \Phi_\infty)$ is indeed a Higgs bundle.

Step 2. Construct nonzero holomorphic map $g_\infty : E \to E_\infty$: Let $g_j(A) = A_j$ be the complex gauge transformations associated to the minimizing sequence (A_j, Φ_j). That is,

$$g_j\Phi = \Phi_j g_j.$$

Using a similar argument as in Step 1, one shows that $g_j \to g_\infty$ in C^α, and that g_∞ is holomorphic with

$$g_\infty\Phi = \Phi_\infty g_\infty.$$

Step 3. The gauge transformation g_∞ is an isomorphism: Suppose g_∞ is not an isomorphism. *Then the existence of the Φ-invariant subbundle $S = \ker g_\infty \subseteq \mathcal{E}$ contradicts our assumption that $\mathcal{E}$ is stable.*

Step 4. Finally, one *checks* that the a priori weak solution (A_∞, Φ_∞) is a solution to Hitchin's equation by checking that it is a critical point of the Yang-Mills-Higgs functional.

Due to time constraints, it was not possible to cover all details of this proof. Please refer to Theorem 2.17 of [Wen16] to fill in the gaps.

□

References

[Wen16] Richard Wentworth. "Higgs bundles and local systems on Riemann surfaces". In: *Advanced Courses in Mathematics - CRM Barcelona* (2016), pp. 165–219. DOI: 10.1007/978-3-319-33578-0_4.