Discussion Sheet: Gauge Theory

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This sheet provides some questions for anchoring, feel free to discuss anything else with your neighbors! The goal is to awaken your working memory of principal bundles and gauge theory, and get to know people sitting next to you. Questions marked with $(*)$ are longer or are curious facts that one should focus on gaining intuition instead of being bogged down. It is more important to consider how theory applies to concrete examples. You can find most answers in Sections 2-3 of Victoria Hoskins notes [Hos13]. For a more comprehensive introduction, see [Ham17], and maybe [Bau09] (in German).

G-principal bundles

1. Recall the definition of a G-principal bundle, find nontrivial examples of:
   - an $\mathrm{U}(1)$-principal bundle,
   - a $\mathrm{GL}(n, \mathbb{C})$-principal bundle,
   - an $\mathrm{O}(n)$-principal bundle.
2. Given a connected manifold X, its universal cover $\tilde{X} \to X$ is a principal bundle of some discrete group G. Which group is it?
3. Discuss the transition function of a G-principal bundle, what should be the correct notion of bundle maps between principal bundles?
4. A principal bundle is trivial if and only if it admits a global section. Can you see why this is true?

Connection and curvature

1. A connection on a G-principal bundle is a $\mathfrak{g}$-valued 1 form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the following:

   (1) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$,

   (2) $\omega_p \circ \rho_p = \mathrm{id}_{\mathfrak{g}}$, where $\rho : P \times \mathfrak{g} \to TP$ denotes the infinitesimal action

   $$\rho_p(\eta) = \frac{d}{dt} p \cdot \exp(t\eta)|_{t=0} \in T_p P.$$

   Figure out what these symbols mean and understand how the connection allows tangent space splits into a vertical and horizontal part $TP = T^v P \oplus T^h P$ by setting $T^h P = \ker \omega$.

   This gives us a way to lift paths $c : [0,1] \to X$ on the base to a unique(!) horizontal path $\tilde{c} : [0,1] \to P$ with a given starting point whose tangent vectors are horizontal. (Why?)

2. Could you see that the difference of two connections is again a connection? This makes the space of connections an affine space $\mathcal{A} \subseteq \Omega^1(P, \mathfrak{g})$.

3. (*)The curvature of a connection ω is a 2-form $F_\omega \in \Omega^2(P, \mathfrak{g})$ given by

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega].$$

A principal bundle is called **flat** if it admits a connection whose curvature is zero. Verify that the Maurer-Cartan connection $\theta \in \Omega^1(P, \mathfrak{g})$ on the trivial G-principal bundle $P = X \times G$, defined by

$$\theta_{(x,g)} : T_pP \cong T_xX \times T_gG \to \mathfrak{g},$$
$$(v, B) \mapsto L_{g^{-1}*}(B).$$

is a well-defined connection that is indeed flat.

4. (*)**Fact:** A flat bundle is not necessarily trivial, conversely a trivial bundle can admit non-flat connections. Can you come up with some intuition as to why this is true?

Associated vector bundles

1. To a G-principal bundle P and a representation $\rho : G \to \mathrm{GL}(V)$, we define an **associated vector bundle** $P \times_G V$,

$$P \times_G V = P \times V / \sim_G$$

where $\sim_G$ is the G-action $g \cdot (p, v) \sim (p \cdot g, \rho(g^{-1}) \cdot v)$. Can you figure out the transition functions of the associated vector bundle given that of P?

2. There is an equivalence of categories between rank n complex vector bundles and $\mathrm{GL}(n, \mathbb{C})$-principal bundles over a common base X. Can you figure out the bijection (not the categorial equivalence!) using the framework of associated vector bundles? In particular, can you see that the tangent bundle TX of a manifold X is the associated vector bundle of the frame bundle $\mathrm{Fr}(X)$ over X?

3. (*)A connection ω on P induces an affine connection ∇ on $P \times_G V$. Can you figure out how this works?

4. (*)**Fact:** Let P be a $\mathrm{GL}(n, \mathbb{C})$-principal bundle and $E := P \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n$ the associated vector bundle via the standard representation. In this case, the **adjoint bundle** $\mathrm{Ad}(P) := P \times_{\mathrm{GL(n,\mathbb{C})}} \mathfrak{gl}(n, \mathbb{C})$ is isomorphic to $\mathrm{End}(E)$. Why this is true?

References

[Bau09] Helga Baum. *Eichfeldtheorie*. Springer, 2009.

[Ham17] Mark Hamilton. *Mathematical gauge theory*. Springer, 2017.

[Hos13] Victoria Hoskins. *On algebraic aspects of the moduli space of flat connections*. 2013. URL: `https://www.math.ru.nl/~vhoskins/talk_connections.pdf`.