Discussion sheet: Proj and Ample line bundles

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We want to associate to an integral graded finitely generated $\mathbb{C}$-algebra $A = \bigoplus_{n\geq 0} A_n$ a projective variety $\operatorname{Proj} A$. As a warm-up, we discuss some properties of the Spec construction and we will see later to what extend they also hold for Proj.

1 The affine spectrum

We recall some facts about affine varieties. Given an integral finitely generated $\mathbb{C}$-algebra B, we have associated a affine variety $\operatorname{Spec} B$. Moreover, if B' is another integral finitely generated $\mathbb{C}$-algebra, then we have a natural bijection

$$\operatorname{Hom}_{\mathbb{C}\text{-alg}}(B, B') \cong \operatorname{Hom}_{\mathbb{C}}(\operatorname{Spec} B', \operatorname{Spec} B) \tag{1}$$

In other words, the categories of integral finitely generated $\mathbb{C}$-algebras and the categories of affine varieties are contravariantly equivalent.

1.1 The underlying set

For a finitely generated $\mathbb{C}$-algebra B, we considered

$$\operatorname{mSpec} B = \{\, \mathfrak{m} \subset B \mid \mathfrak{m} \text{ is a maximal ideal} \,\}. \tag{2}$$

In the modern scheme theoretic foundation, the underlying space of the variety associated to B is

$$\operatorname{Spec} B = \{\, \mathfrak{p} \subset B \mid \mathfrak{p} \text{ is a prime ideal} \,\}. \tag{3}$$

Recall that an ideal $\mathfrak{p}$ is prime iff for $a, b \in B$ we have that if the product ab is in $\mathfrak{p}$, then one of the factors a or b has to be in $\mathfrak{p}$. In other words, the quotient $B/\mathfrak{p}$ is integral.

Exercise 1.1. *Show that for $B = \mathbb{C}[x]$ the maximal ideals are of the form $(x - \alpha)$ for $\alpha \in \mathbb{C}$, but (0) is also a prime ideal.*

For $f \in B$, we have $D(f) = \{\, \mathfrak{p} \subset B \mid f \notin \mathfrak{p} \,\}$. These form a basis of the topology on $\operatorname{Spec} B$.

Exercise 1.2. *Give a complete description of the open sets in $\operatorname{Spec} \mathbb{C}[x]$. In particular, show that (0) is contained in every non-empty open set. It is called the "generic" point of $\operatorname{Spec} \mathbb{C}[x]$.*

Exercise 1.3. *One often writes $f(\mathfrak{p}) \neq 0$ instead of $f \notin \mathfrak{p}$. Convince yourself that this notation makes sense by looking at the maximal ideals of $\mathbb{C}[x]$.*

2 The Proj construction

We want to associate to an integral graded finitely generated $\mathbb{C}$-algebra $A = \bigoplus_{r\geq 0} A_r$ a projective variety $\operatorname{Proj} A$. The example that you should have in mind is $\operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n] = \mathbb{CP}^n$. In the modern scheme theoretic setting, one can associate a projective $\mathbb{C}$-scheme $\operatorname{Proj} A$ to a graded finitely generated $\mathbb{C}$-algebra A, so the condition that A is integral is not needed for the general scheme-theoretic construction, but it simplifies the exposition.

2.1 Underlying set

We first define the underlying set of Proj A. In our construction, it will be closer to Spec B than to mSpec B.

First some notations:

- An element $f \in A_r$ is called homogeneous of degree r.
- An ideal $\mathfrak{a} \subset A$ is called graded if it is generated by its homogeneous elements.
- The ideal $A_+ = \bigoplus_{r>0} A_r$ is called the irrelevant ideal.

Definition 2.1. *The underlying space of* Proj A *is given by*

$$\{\, \mathfrak{p} \subset A \mid \mathfrak{p} \text{ graded prime ideal}, A_+ \not\subset \mathfrak{p} \,\}. \tag{4}$$

For $f \in A_+$ homogeneous, we consider the standard open set

$$D_+(f) = \{\, \mathfrak{p} \in \operatorname{Proj} A \mid f \notin \mathfrak{p} \,\} \tag{5}$$

The sets form an basis for the Zariski topology on Proj A.

Exercise 2.1. *Show that $D_+(fg) = D_+(f) \cap D_+(g)$.*

Exercise 2.2 (Optional)**.** *In this exercise we deal with graded $\mathbb{C}$-algebras that are not necessarily integral or satisfy $A_0 = \mathbb{C}$, in order to motivate that (4) is the correct general definition.*

a) Let A be a finitely generated graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$. Show that every homogeneous prime ideal is contained in the irrelevant ideal A_+.

b) If B, B' are $\mathbb{C}$-algebras, the direct product $B \times B'$ is also a $\mathbb{C}$-algebra. The sets $B \times 0$ are $0 \times B'$ are ideals in $B \times B'$, and every prime ideal $\mathfrak{p} \subset B \times B'$ contains exactly one of $\{B \times 0, 0 \times B'\}$.

c) Show that for A, A' finitely generated graded $\mathbb{C}$-algebras, show that we have an homeomorphism $\operatorname{Proj}(A \times A') \cong \operatorname{Proj} A \sqcup \operatorname{Proj} A'$, *where $A \times A'$ has the grading $(A \times A')_n = A_n \times A'_n$.*

2.2 Affine charts

From now on we again assume for simplicity that A is integral. All fractions that will appear can be understood in the quotient field of A. For $f \in A_r$ non-zero for $r > 0$, we define a ring

$$A_{(f)} = \left\{ \left. \frac{g}{f^k} \,\right|\, k \geq 0, g \in A_{rk} \right\} \tag{6}$$

Exercise 2.3. *Let f_1, f_2 be non-zero, homogeneous of positive degree. Show that $A_{(f_1)} \subset A_{(f_1 f_2)}$ as subsets of the quotient field of A.*

Theorem 2.1. *The space* Proj A *carries the structure of a projective variety such that for every f homogeneous non-zero of positive degree, $D_+(f) \cong \operatorname{Spec} A_{(f)}$. We have the following compatibility: for f_1, f_2 non-zero, homogeneous of positive degree, we have that the inclusion $D_+(f_1 f_2) \subset D_+(f_1)$ is the morphism contravariantly equivalent to the inclusion $A_{(f_1)} \subset A_{(f_1 f_2)}$.*

Exercise 2.4. *For $A = \mathbb{C}[x_0, x_1]$, use the covering* $\operatorname{Proj} A = D_+(x_0) \cup D_+(x_1)$ *to convince yourself that* $\operatorname{Proj} A = \mathbb{CP}^1$.

a) Show that $\operatorname{Proj} A = D_+(x_0) \cup D_+(x_1)$ *is indeed a covering (Hint: $A_0 = \mathbb{C}$ and x_0, x_1 generate the irrelevant ideal.)*

b) Show $A_{(x_0)} = \mathbb{C}[\frac{x_1}{x_0}], A_{(x_1)} = \mathbb{C}[\frac{x_0}{x_1}], A_{(x_0 x_1)} = \mathbb{C}[\frac{x_1}{x_0}, \frac{x_0}{x_1}]$.

c) Interpret the previous calculations as the description of Proj A *as two copies of $\mathbb{A}^1$ glued along $\mathbb{A}^1 \setminus \{0\}$ along the identification $\eta = t^{-1}$.*

2.3 Proj as a functor

In this section $A \subset A'$ are two finitely generated graded $\mathbb{C}$-algebras (with the same grading, so $A_r \subset A'_r$ for all $r \geq 0$).

Exercise 2.5. *If $B \subset B'$ are two $\mathbb{C}$-algebras, and $\mathfrak{p} \subset B'$ is a prime ideal, show that $\mathfrak{p} \cap B$ is a prime ideal. Note: this is also called the* contraction *of $\mathfrak{p}$ along the inclusion. In fact, this is the set-theoretic map $\operatorname{Spec} B' \to \operatorname{Spec} B$ associated to the inclusion.*

Exercise 2.6. *If $\mathfrak{a} \subset A'$ is a graded ideal, then $\mathfrak{a} \cap A$ is a graded ideal.*

With these exercises we could try to define a map from $\operatorname{Proj} A'$ to $\operatorname{Proj} A'$ via contraction. It is not necessarily globally defined:

Exercise 2.7. *If $A = \mathbb{C}[x_0, x_1] \subset A' = \mathbb{C}[x_0, x_1, x_2]$, show that the ideal generated by x_0 and x_1 in A' gives an element in $\operatorname{Proj} A'$, but its contraction to A is $A_+ \notin \operatorname{Proj} A$.*

Exercise 2.8. *For $A \subset A'$ inclusion of integral finitely generated graded $\mathbb{C}$-algebras, show that*

$$U := \{ \mathfrak{p} \in \operatorname{Proj} A' \mid A_+ \not\subset \mathfrak{p} \} \tag{7}$$

is a (dense) open subset with respect to the Zariski topology.

Theorem 2.2. *The map*

$$U \xrightarrow{\psi} \operatorname{Proj} A$$
$$\mathfrak{p} \mapsto \mathfrak{p} \cap A$$

is the underlying map of a morphism of varieties. Moreover if $f \in A_+$ is non-zero homogeneous, $\psi^{-1}(\operatorname{Spec} A_{(f)}) = \operatorname{Spec} B_{(f)}$, and ψ restricts to the morphism associated to the inclusion $A_{(f)} \subset B_{(f)}$.

A morphism defined on a dense open subset of a variety is also called a *rational* morphism, we denote this by $\operatorname{Proj} A' \dashrightarrow \operatorname{Proj} A$.

Exercise 2.9. *What is the geometric meaning of the map associated to the inclusion $\mathbb{C}[x_0, x_1] \subset \mathbb{C}[x_0, x_1, x_2]$?*

Exercise 2.10. *Let $A = \bigoplus_{r \geq 0} A_r$ be a finitely generated graded $\mathbb{C}$-algebra. Consider $A^{(d)} = \bigoplus_{r \geq 0} A_{dr}$.*

a) Show that $A^{(d)}$ is a graded subalgebra of A.

b) Show that $\mathfrak{p} \mapsto \mathfrak{p} \cap A^{(d)}$ defines a homeomorphism of $\operatorname{Proj} A$ and $\operatorname{Proj} A^{(d)}$.

The map is in fact the underlying map of an isomorphism of varieties.

3 Line bundles

3.1 Vector bundles as locally free sheaves

Similar to vector bundles over smooth manifolds, one can define vector bundles of rank r over a variety X as a map $\pi : E \to X$ over a variety X where every fibre over a (closed) point $x \in X$ is endowed with a $\mathbb{C}$-vector space structure, and we have an affine cover $(U_i)_{i \in I}$ of X such that the vector bundle is locally trivial, i.e., $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^r$. Given a vector bundle, we can consider its sheaf of sections

$$\Gamma(U, E) := \{ s \mid U \to E \colon \pi \circ s = \mathrm{id}_U \} \tag{8}$$

Exercise 3.1. *Let U be a trivializing affine on X, so that $\phi\colon \pi^{-1}(U) \to U \times \mathbb{A}^r$ is the trivialization. Let $t_i\colon \mathbb{A}^r \to \mathbb{A}^1$ be the projection to the i-th component.*

Show that

$$\Gamma(U, E) \to \mathcal{O}_X(U)^r \tag{9}$$
$$s \mapsto (t_1 \circ \phi \circ s, \ldots, t_r \circ \phi \circ s) \tag{10}$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

The exercise shows that the sheaf of sections of a vector bundle is locally free. In fact, we have the following:

Theorem 3.1. *There is a equivalence of categories between vector bundles and locally free sheaves.*

We are in particular interested in line bundles, i.e., locally free sheaves of rank 1.

3.2 Twisting sheaf

Let us now consider the following

Definition 3.1. *For $d \in \mathbb{Z}, U \subset \mathbb{CP}^n$ Zariski open, define*

$$\mathcal{O}_{\mathbb{CP}^n}(d)(U) = \{ \frac{g}{f} \mid f \in \mathbb{C}[x_0, \dots, x_n]_e, g \in \mathbb{C}[x_0, \dots, x_n]_{e+d} \text{ for some } e \in \mathbb{Z}, f(p) \neq 0 \quad \forall p \in U \} \tag{11}$$

Note that the sets $\mathcal{O}_{\mathbb{CP}^n}(d)(U)$ are always subsets of the field $\mathbb{C}(x_0, \dots, x_n)$.

Theorem 3.2. *For every d, $\mathcal{O}_{\mathbb{CP}^n}(d)$ defines a locally free sheaf associated to a line bundle on $\mathbb{CP}^n$.*

Let us unpack this for $n = 1$

Exercise 3.2. *a) Show that $\mathcal{O}_{\mathbb{CP}^1}(d)(D_+(x_0)) = x_0^d \mathbb{C}[\frac{x_1}{x_0}], \mathcal{O}_{\mathbb{CP}^1}(d)(D_+(x_1)) = x_1^d \mathbb{C}[\frac{x_0}{x_1}]$ and deduce that one can trivialize $\mathcal{O}_{\mathbb{CP}^1}(d)$ over $D_+(x_0)$ and $D_+(x_1)$.*

b) Show that $\mathcal{O}_{\mathbb{CP}^1}(d)(\mathbb{CP}^1) = \mathbb{C}[x_0, x_1]_d$. In particular, $\mathcal{O}_{\mathbb{CP}^1}(d)(\mathbb{CP}^1) = 0$ for $d < 0$, $\mathcal{O}_{\mathbb{CP}^1}(0)(\mathbb{CP}^1) = \mathbb{C}$.

c) Do similar computations for $\mathbb{CP}^n$.

3.3 Linear series

Let X be a projective variety. Suppose $\mathcal{L}$ is a line bundle on X. We can consider $\Gamma(X, \mathcal{L})$, the global sections of $\mathcal{L}$. This is a finite dimensional vector space. We can define a rational morphism from X to $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ as follows:

Let $x \in X$ be a (closed) point. If $\pi : \mathcal{L} \to X$ is the projection map (thinking of L as an actual vector bundle), we can take an identification $\pi^{-1}(x) \cong \mathbb{C}$. The identification is unique up to a scalar choice $\in \mathbb{C}^*$. We can then consider the map $\Gamma(X, \mathcal{L}) \xrightarrow{ev_x} \pi^{-1}(x) \cong \mathbb{C}$. If not all global sections vanish at x, then this gives us an element in $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$.

Definition 3.2. *We say that $\mathcal{L}$ is very ample if the above map is globally defined and is a closed immersion. A line bundle is ample if some tensor power is very ample.*

Exercise 3.3. *Check that the above construction gives the identity under appropriate identifications for $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^1}(1)$.*

Closely related, for a projective variety X and a line bundle $\mathcal{L}$, we can consider the graded ring $R(X, \mathcal{L}) = \bigoplus_{r \geq 0} \Gamma(X, \mathcal{L}^{\otimes r})$. There is a rational morphism $X \dashrightarrow \operatorname{Proj} R(X, \mathcal{L})$, sending a point x to the ideal of vanishing sections at x. In fact, there is the following alternative description for ample: On a projective variety X, a line bundle $\mathcal{L}$ is ample if and only if the associated morphism $X \dashrightarrow \operatorname{Proj} R(X, \mathcal{L})$ is an isomorphism.