Discussion sheet: Rank one case

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In this sheet, we discuss the nonabelian Hodge correspondence in the case $G = \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$. This is an abelian complex Lie group. We will derive the moduli spaces and examine the correspondence between them. A good source that worked out the rank one case using the language of deformation theory and groupoids is [GX08].

Let Σ be a closed orientable surface with genus $g \geq 2$, and fix $L \to \Sigma$ be a complex line bundle of degree 0. Furthermore, since line bundles on curves are classified by their degree, this means that up to smooth bundle isomorphism, L is the trivial line bundle over Σ. Here, our choice of $G = \mathrm{GL}(1, \mathbb{C})$ appears as the structure group of L.

The Betti moduli space

By definition, the Betti moduli space is the affine GIT quotient

$$\mathcal{M}_B := \mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^\times) /\!\!/ \mathbb{C}^\times,$$

where $\mathbb{C}^\times$ acts by conjugation. Using the fact that $\mathbb{C}^\times$ is abelian, deduce that $\mathcal{M}_B = (\mathbb{C}^\times)^{2g}$ with a natural complex structure induced by that of $\mathbb{C}^\times$.

The de Rham moduli space

Let $\mathcal{A}_L$ denote the affine space of connections on $L \to \Sigma$ modeled after the vector space of endomorphism valued 1-forms $\Omega^1(\Sigma, \mathrm{End}(L))$, where $\mathcal{A}_L^{flat} \subseteq \mathcal{A}_L$ denote the space of flat connections. Let $\mathcal{G}_L := \Omega^0(\Sigma, \mathrm{Aut}(L))$ be the gauge group acting on $\mathcal{A}_L$ by conjugation.

By definition, the de Rham moduli space is the quotient

$$\mathcal{M}_{dR} := \mathcal{A}_L^{flat} / \mathcal{G}_L.$$

Deduce that

$$\mathcal{M}_{dR}(\Sigma) = H^1(\Sigma, \mathbb{C}) / H^1(\Sigma, \mathbb{Z}) = \mathbb{C}^{2g} / \mathbb{Z}^{2g}$$

by going through the following steps:

(i) Show that $\mathcal{A}_L^{flat}$ is an affine space over the closed one-forms $Z^1(\Sigma) := \{\eta \in \Omega^1(\Sigma) \,|\, d\eta = 0\}$ and that $\mathcal{G}_L$ acts on $\mathcal{A}_L^{\mathrm{flat}}$ by $g \cdot D = d + g(dg^{-1}) + \eta$ for $D = d + \eta$. Convince yourself that $g \cdot D$ is again flat.

(ii) Let $\mathcal{G}_0 \subseteq \mathcal{G}_L$ denote the component of null homotopic maps. Show that any null homotopic map $g : \Sigma \to \mathbb{C}^*$ is $g = \exp f$ for some $f : \Sigma \to \mathbb{C}$, and that $\mathcal{A}_L^{flat} / \mathcal{G}_0 = H^1(\Sigma, \mathbb{C})$.

(iv) Show that $\pi_0(\mathcal{G}_L) \cong H^1(\Sigma, \mathbb{Z})$ by using that $g \in \mathcal{G}_L$ induces a group homomorphism $g_* \in \mathrm{Hom}(\pi_1(\Sigma), \pi_1(\mathbb{C}^\times) = \mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z})$.

(v) Conclude the statement by using $\mathcal{A}_L^{flat}/\mathcal{G}_L = (\mathcal{A}_L^{flat}/\mathcal{G}_0)/\pi_0(\mathcal{G}_L)$.

(vi) Show that the complex structure on $\mathcal{M}_{dR}$ agrees with that on $\mathcal{M}_B$.

The Dolbeault moduli space

Fix a complex structure on Σ to obtain $X := (\Sigma, \overline{\partial})$ a compact Riemann surface. Recall that the Dolbeault moduli space is defined to be the space of polystable Higgs bundles

$$\mathcal{M}_{Dol} := \{(L, \overline{\partial}_L, \Phi) \text{ polystable}\}/\sim_{iso},$$

where $(L, \overline{\partial}_L)$ is a holomorphic line bundle over X with underlying smooth bundle L, and $\Phi \in H^0(X, \mathrm{End}(L, \overline{\partial}_L)K)$ is a holomorphic section $L \to LK$, where $K := T^*X$ is the canonical line bundle.

Deduce that

$$\mathcal{M}_{Dol} = \mathrm{Jac}(X) \times \mathcal{H}^{0,1}(X) = (\mathbb{C}^g/\mathbb{Z}^{2g}) \times \mathbb{C}^g,$$

by going through the following steps:

(i) Show that every rank one Higgs bundle is stable, hence $\mathcal{M}_{Dol} := \{(L, \overline{\partial}_L, \Phi)\}/\sim_{iso}$.

(ii) The isomorphism classes of degree 0 holomorphic line bundles over X is called the **Jacobian**, $\mathrm{Jac}(X) \subseteq \mathrm{Pic}(X)$, and is the kernel of map c_1,

$$\mathrm{Jac}(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}),$$
$$[(L, \overline{\partial}_L)] \mapsto c_1(L),$$

induced by the long exact cohomology sequence of the exponential sequence $\mathbb{Z} \to \mathcal{O} \to \mathcal{O}^*$.

(iii) Given a holomorphic line bundle $(L, \overline{\partial}_L)$, show that the space of Higgs fields on L is equal to the harmonic $(1,0)$-forms $\mathcal{H}_h^{1,0}(X) = H^0(X, K)$ with respect to a suitable Hermitian metric h.

(iv) Finally, using the fact that the Jacobian is a g-torus

$$\mathrm{Jac}(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g},$$

and that on the compact Riemann surface X, we have the following decomposition:

$$H^1(X, \mathbb{C}) = \mathcal{H}_h^{1,0}(X) \oplus \mathcal{H}_h^{0,1}(X)$$

where $\mathcal{H}_h^{1,0}(X) = \overline{\mathcal{H}_h^{0,1}}(X)$, deduce our desired result.

(v) What is the complex structure on $\mathcal{M}_{Dol}$ induced from the complex structure on X? How does it differ from the complex structure on $\mathcal{M}_B$?

References

[GX08] William M. Goldman and Eugene Z. Xia. "Rank one higgs bundles and representations of fundamental groups of Riemann surfaces". In: *Memoirs of the American Mathematical Society* 193.904 (2008), 0–0. DOI: 10.1090/memo/0904.