Fun with Reynolds operators

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This note is entirely based on Derksen & Kemper's *Computational Invariant theory* [1], specifically Chapter 4.5, with the potential introduction of typos to the exposition.

1 From reductive groups to semisimple ones

Let X be an affine variety, $R \coloneqq K[X]$ and let G be a linear algebraic group acting on X. A Reynolds operator is a K-linear map $\mathcal{R}\colon R \to R^G$ such that

(a) (Idempotence) $\mathcal{R}(f) = f$ for R^G

(b) (G-invariance) $\mathcal{R}(\sigma \cdot f) = \mathcal{R}(f)$ for $f \in R$, $\sigma \in G$.

Since G acts trivially on R^G (by definition), the second condition is equivalent to G-equivariance ($\mathcal{R}(\sigma \cdot f) = \sigma \cdot \mathcal{R}(f)$), so all we really require is a split of the inclusion map $R^G \hookrightarrow R$ that is a map of representation.

If the G-module R is completely reducible (always the case if G is linearly reductive), then there is *unique* such split: The projection to the isotypic component

$$R = R^G \oplus \bigoplus_{\rho \neq 1} R^\rho.$$

Example 1.1. If G is finite and $\operatorname{char} K \nmid \#G$, then the Reynolds operator is

$$\mathcal{R}(f) = \frac{1}{\#G} \sum_{\sigma \in G} \sigma \cdot f.$$

The proof is an immediate application of Schur's lemma. ♢

In principle, this can be extended to arbitrary reductive groups over $\mathbb{C}$: There always exists a maximal compact subgroup $C \subseteq G$ which is Zariski-dense; then

$$\mathcal{R}(f) = \int_C \sigma \cdot f \,\mathrm{d}\mu(\sigma),$$

where μ is the normalized right Haar measure on C.

We can reduce the construction of Reynolds operators on coordinate rings of affine G-varieties to the construction of Reynolds operators on the G-variety $X = G$. Before we do this, let's look at an example

Example 1.2. Let $G = T = (K^\times)^n$ be the torus. In the case $K = \mathbb{C}$, a maximal compact subgroup is given by $K = (\mathbb{S}^1)^n$, and integration over K amounts to averaging. $K[T] = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is the Laurent monomial ring, and for any monomial z^m we have

$$\int_{(\mathbb{S}^1)^n} z^m \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \frac{\mathrm{d}z_n}{z_n} = \prod_{i=1}^n \int_{\mathbb{S}^1} z_i^{m_i} \frac{\mathrm{d}z_i}{z_i} = \prod_{i=1}^n \delta_{m_i,0}.$$

Let's extend this to general fields and varieties:

An action of T on X induces a $\mathbb{Z}^n$-grading on R:

$$R = \bigoplus_{m \in \mathbb{Z}^n} R^{\chi_m}, \qquad \sigma \cdot f = \chi_m(\sigma) \cdot f \text{ for } \sigma \in T,\ f \in R^{\chi_m}.$$

For $K^\times$ this is just a $\mathbb{Z}$-grading and these are the graded components. Then the Reynolds operator sends f to its degree 0 component ($0 \in \mathbb{Z}^n$). Put differently, $T \times X \to X$ induces $R \to K[T \times X] \cong R[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, and the isotypic decomposition is given by extracting the coefficient of z^χ. $\mathcal{R}(f)$ is then the $z^0 = 1$-coefficient of f. ♢

If $N \trianglelefteq G$ is a normal subgroup, then we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\mathcal{R}_N"'] \arrow[rr, bend left, "\mathcal{R}_G"] & R^N \arrow[r, "\mathcal{R}_{G/N}"'] & R^G. \end{tikzcd}$$

This allows to reduce the study of reductive algebraic groups to that of connected semisimple groups:

- If G° is the connected component of e_G, then $G^\circ \trianglelefteq G$ and G/G° is finite ($\rightsquigarrow$ average).
- If G is connected reductive, then $G/[G,G]$ is abelian and reductive, hence a torus ($\rightsquigarrow$ multigraded approach), and $[G,G]$ is semisimple.

Recall that a connected linear group G is semisimple if

$$R(G) \coloneqq \bigcap_{B \subseteq G \text{ Borel}} B = \{e\},$$

where a Borel subgroup is a maximal connected solvable group.

2 The dual space $K[G]^*$ and the Lie algebra $\mathfrak{g}$

The dual space $K[G]^*$ is home to many of our friends

- Most importantly, since $K[G]^G = K$, the Reynolds operator of G is an element $\mathcal{R}_G \in K[G]^*$
- Every element $\sigma \in G$ induces via evaluation a functional $\epsilon_\sigma \colon K[G] \to K$, $\epsilon_\sigma(f) = f(\sigma)$. The Reynolds property can be restated as $\mathcal{R}_G * \epsilon_\sigma = \mathcal{R}_G$.

- $K[G]^*$ is a associative algebra with identity ϵ_e and product

$$\delta * \gamma := \cdot \circ (\delta * \gamma) \circ m^* \colon K[G] \to K[G] \otimes K[G] \to K \otimes K \to K$$
$$(\delta * \gamma)(f) = \sum_i \delta(g_i)\gamma(h_i), \qquad m^*(f) = \sum_i g_i \otimes h_i.$$

We can extend the action of G on $K[X]$ to an action of $\delta \in K[G]^*$ on $R = K[X]$ via $(\delta \otimes \mathrm{id}_{K[X]}) \circ \mu^*$:

$$(\delta \cdot f)(x) = \delta(\sigma \mapsto f(g \cdot x)).$$

In particular, $\mathcal{R}_G$ can be seen as a linear operator

$$\mathcal{R}_X \colon R \to R, \qquad f \mapsto \mathcal{R}_G \cdot f.$$

We verify that $\mathcal{R}_X$ is indeed a Reynolds operator on R:

Proof. 1. If $f \in R^G$, then $\mu^*(f) = f \otimes 1$, so $\mathcal{R}_X(f) = \mathcal{R}_G(1) \cdot f = f$.

2. Since $\mathcal{R}_G * \epsilon_\sigma = \mathcal{R}_G$, we have

$$\mathcal{R}_X(\sigma \cdot f) = \mathcal{R}_G \cdot (\sigma \cdot f) = \mathcal{R}_G \cdot (\epsilon_\sigma \cdot f) = (\mathcal{R}_G \cdot \epsilon_\sigma) \cdot f = \mathcal{R}_G \cdot f = \mathcal{R}_X(f). \quad \square$$

The Lie algebra $\mathfrak{g}$ of G can be identified with the subspace

$$T_e(G) = (\mathfrak{m}_e/\mathfrak{m}_e^2)^* \cong \{\, \delta \mid \delta(fg) = \delta(f)g(e) + f(e)\delta(g) \ \forall f, g \in K[G] \,\} \subseteq K[G]^*$$
$$\delta \mapsto (f \mapsto \delta(f) - f(e) + \mathfrak{m}_e^2).$$

The Lie bracket is given by $[\delta, \gamma] = \delta * \gamma - \gamma * \delta$.

For $\delta \in \mathfrak{g}$, define the adjoint element $\mathrm{ad}(\delta) \in \mathrm{End}_K(\mathfrak{g})$ by $\mathrm{ad}(\delta)(\gamma) = [\delta, \gamma]$. The symmetric bilinear *Killing form* $\kappa \in S^2(\mathfrak{g})$ is

$$\kappa(\delta, \gamma) = \mathrm{tr}(\mathrm{ad}(\delta) \circ \mathrm{ad}(\gamma) \colon \mathfrak{g} \to \mathfrak{g}).$$

$\mathfrak{g}$ is semisimple iff κ is non-degenerate (this requires characteristic 0). Let $\delta_1, \ldots, \delta_m \in \mathfrak{g}$ be a basis with κ-dual basis $\delta^1, \ldots, \delta^m \in \mathfrak{g}$. Then the Casimir operator is

$$c = \sum_{i=1}^m \delta_i * \delta^i \in K[G]^*.$$

Lemma 2.1. *1. The Killing form G-invariant: $\kappa(\sigma \cdot \delta, \sigma \cdot \gamma) = \kappa(\delta, \gamma)$. Any G-invariant symmetric bilinear form is a scalar multiple of κ.*

2. The Casimir operator commutes with all evaluation functionals: $\epsilon_\sigma * c = c * \epsilon_\sigma$.

3. In fact, it commutes with the action of G on any rational representation, hence acts as a scalar on every irrep V. This scalar is 0 if and only if V is trivial.

Example 2.2. $G = \mathrm{GL}_n$, $\mathfrak{g} = \mathfrak{gl}_n$ is isomorphic to $K^{n\times n}$ as follows: Every matrix A defines a directional derivative

$$\partial_A f \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} f(I + tA)|_{t=0}, \qquad f \in K[\mathrm{GL}_n]$$

In this case $[\delta_A, \delta_B] = \delta_{[A,B]}$.

Similarly, $G = \mathrm{SL}_n$ has the traceless matrices $\mathfrak{sl}_n$ as its Lie algebra. A basis is given by $\partial_{i,j}$, $i \neq j$, and $\partial_{i,i} - \partial_{i+1,i+1}$. For $\mathfrak{sl}_n$, its Killing form is given by $\kappa(A, B) = 2\operatorname{tr}(AB)$. The dual basis is given by $\partial_{j,i}$ and $\frac{n-i}{n}(\partial_{1,1} + \cdots + \partial_{i,i}) - \frac{i}{n}(\partial_{i+1,i+1} + \cdots + \partial_{n,n})$. The Casimir operator is given by

$$2n \cdot c = \sum_{i,j} \partial_{i,j} * \partial_{j,i} - \frac{1}{n} \sum_{i,j} \partial_{i,i} * \partial_{j,j}. \qquad \diamondsuit$$

3 Computing Reynolds using Casimir

We can use the Casimir operator to compute the Reynolds operator $\mathcal{R}\colon V \to V^G$ on a rational representation V as follows:

Proposition 3.1. *Let V be a rational representation of G and $v \in V$. Let $P = \sum_i a_i t^i \in K[t]$ be the monic polynomial of smallest degree such that $P(c) \cdot v = 0$, then*

$$\mathcal{R}(v) = \begin{cases} 0 & \text{if } P(0) \neq 0 \\ Q(0)^{-1} Q(c) \cdot v & \text{if } P(t) = tQ(t). \end{cases}$$

Note that P exists since a rational representation V is the union of finite-dimensional subrepresentations (which are all preserved by c).

Proof. Write $v = v_0 + v_1$, $v_0 \in V^G$, $v_1 \in C$, then $\mathcal{R}(v) = v_0$ and, since c acts by scaling on irreps,

$$0 = P(c)v = P(c)v_0 + P(c)v_1 \in V^G \oplus C \qquad \rightsquigarrow \qquad P(c)v_0 = P(c)v_1 = 0.$$

Since c acts on V^G as the zero map, $P(0)v_0 = P(c)v_0 = 0$, so $v_0 = 0$ if $P(0) \neq 0$.

On the other hand, if $P(0) = 0$, then $0 = P(c)v_1 = c(Q(c)v_1)$, by injectivity of c on C, $Q(c)v_1 = 0$. On the other hand, $Q(c)v_0 = Q(0)v_0$ and $Q(c)v = Q(0)v_0$. By minimality of $\deg P$, $Q(0) \neq 0$. $\square$

This leads to a simple algorithm to compute $\mathcal{R}(v)$: For $l = 0, 1, \ldots$, check if there is a linear combination $\sum_{i=0}^{l} a_i v_i = 0$, where $v_i = c^{*i} \cdot v_0$. Once found, output 0 if $a_0 \neq 0$, otherwise $a_1^{-1} \sum_{i=1}^{l} a_i v_{i-1}$. One can make the linear independence test more efficient using leading terms with respect to a term order.

Example 3.2. For $G = \mathrm{SL}_2$, a basis of $\mathfrak{g} = \mathfrak{sl}_2$ is given by

$$\boldsymbol{x} = \partial_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{x} = \partial_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{h} = \partial_{1,1} - \partial_{2,2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The Killing form and hence the Casimir element in this basis are given by

$$\kappa = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \qquad c = \frac{\boldsymbol{x} * \boldsymbol{y}}{4} + \frac{\boldsymbol{y} * \boldsymbol{x}}{4} + \frac{\boldsymbol{h} * \boldsymbol{h}}{8}.$$

The action of this basis on $v \in V = \mathbb{C}[s,t]_2 = \left\{ a_0 s^2 + a_1 st + a_2 t^2 \;\middle|\; a \in K^3 \right\}$ and the coordinate ring $f \in K[V] = K[a_0, a_1, a_2]$ is given by

$$\begin{array}{lll} \boldsymbol{x} \cdot v = y\partial_x v & \boldsymbol{y} \cdot v = x\partial_y v & \boldsymbol{h} \cdot v = x\partial_x v - y\partial_y v \\ \boldsymbol{x} \cdot f = -2a_0\partial_{a_1} f - a_1\partial_{a_2} f & \boldsymbol{y} \cdot f = -a_1\partial_{a_0} f - 2a_2\partial_{a_1} f & \boldsymbol{h} \cdot f = -2a_0\partial_{a_0} f + 2a_2\partial_{a_2} f. \end{array}$$

For example, let $f = a_1^2$, then

$$c \cdot f = 2a_1^2 + 4a_0a_2, \qquad c^2 \cdot f = 6a_1^2 + 12a_0a_2.$$

f and $c \cdot f$ are linearly independent, but $(c^2 - 3c) \cdot f = 0$, so Proposition 3.1 tells us that

$$\mathcal{R}(a_0^2) = \frac{1}{-3}(c-3) \cdot f = \frac{2a_1^2 + 4a_0a_2 - 3a_1^2}{-3} = \frac{1}{3}(a_1^2 - 4a_0a_2). \qquad \diamondsuit$$

4 Cayley's Ω-process

Let $\operatorname{char} K =$. In the case $G = \mathrm{GL}_n$ (which is reductive but not semisimple), there is a more direct way to evaluate the Reynolds operator. Let $Z = [z_{ij}]_{i,j=1}^n$, then the coordinate ring is the localization $K[\mathrm{GL}_n] = K[\underline{z}]_{\det Z}$. Consider the differential operator

$$\Omega \coloneqq \det \begin{bmatrix} \partial_{z_{11}} & \dots & \partial_{z_{1n}} \\ \vdots & \ddots & \vdots \\ \partial_{z_{n1}} & \dots & \partial_{z_{nn}} \end{bmatrix} : K[\mathrm{GL}_n] \to K[\mathrm{GL}_n].$$

For example, applying Ω^p to $(\det Z)^p$, we obtain the positive constants

$$c_{p,n} \coloneqq \Omega^p((\det Z)^p) = \prod_{j=1}^{n-1} \frac{(p+j)!}{j!}.$$

Proposition 4.1. *Write* $\frac{f}{(\det Z)^p} \in K[\mathrm{GL}_n]$, $f \in K[\underline{z}]$, *then*

$$\mathcal{R}(f) = \begin{cases} \frac{1}{c_{p,n}} \Omega^{np}(f) & f \textit{ homogeneous of degree } np \\ 0 & \textit{otherwise.} \end{cases}$$

Similar formulae exist for the Reynolds operator associated to SL_n.

References

[1] Harm Derksen and Gregor Kemper. *Computational Invariant Theory.* Springer Berlin Heidelberg, 2015. ISBN: 9783662484227. DOI: 10.1007/978-3-662-48422-7.