Hilbert Functions of Chopped Ideals

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Abstract

A chopped ideal is obtained from a homogeneous ideal by considering only the generators of a fixed degree. We investigate cases in which the chopped ideal defines the same finite set of points as the original one-dimensional ideal. The complexity of computing these points from the chopped ideal is governed by the Hilbert function and regularity. We conjecture values for these invariants and prove them in many cases. We show that our conjecture is of practical relevance for symmetric tensor decomposition.

1 Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0, and let $S = \Bbbk[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables with coefficients in $\Bbbk$. With its standard grading, S is the homogeneous coordinate ring of the n-dimensional projective space $\mathbb{P}^n = \mathbb{P}^n_\Bbbk$. For a tuple of r points $Z = (z_1, \dots, z_r) \in (\mathbb{P}^n)^r$, let $I(Z)$ be the associated vanishing ideal, that is

$$I(Z) \;=\; \langle \{\, f \in S \text{ homogeneous} \mid f(z_i) = 0,\, i = 1, \dots, r \,\} \rangle_S.$$

If the set of points $Z \in (\mathbb{P}^n)^r$ is general, the Hilbert function $h_{S/I(Z)} \colon t \mapsto \dim_\Bbbk (S/I(Z))_t$ is

$$h_{S/I(Z)}(t) \;=\; \min \{ h_S(t) \,,\, r \} \tag{1}$$

where $h_S(t) = \binom{n+t}{n}$ is the Hilbert function of the polynomial ring S. Here, the word *general* means that (1) holds for all Z in a dense Zariski open subset of $(\mathbb{P}^n)^r$. We also have, for general Z, that the ideal $I(Z)$ is generated in degrees d and $d+1$, where d is the smallest integer such that the minimum in (1) equals r [IK99, Thm. 1.69].

This work focuses on a modification $I_{\langle d \rangle}$ of the ideal $I(Z)$, called its *chopped ideal* in degree d. This is defined as the ideal generated by the homogeneous component $I(Z)_d$. In particular $I_{\langle d \rangle} \subseteq I(Z)$, and strict inclusion holds if and only if $I(Z)$ has generators in degree $d+1$. An elementary dimension count shows that there is a range for r for which this happens, see (4).

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If $r < h_S(d) - n$ and Z is general, the saturation $(I_{\langle d\rangle})^{\mathrm{sat}}$ of the chopped ideal with respect to the irrelevant ideal of S coincides with the ideal $I(Z)$. This is proved in Theorem 2.2. In other words, $I(Z)$ and $I_{\langle d\rangle}$ both cut out Z scheme-theoretically. In particular, $I(Z)$ and $I_{\langle d\rangle}$ coincide in large degrees, and they have the same constant Hilbert polynomial, equal to r.

In cases where $(I_{\langle d\rangle})^{\mathrm{sat}} = I(Z)$ and $I_{\langle d\rangle} \neq I(Z)$, there is a range of degrees $d < t < d + e$ with the property that $h_{S/I_{\langle d\rangle}}(t) > h_{S/I(Z)}(t)$. The goal of this work is to determine this *saturation gap*, and understand the geometric and algebraic properties that control it. We illustrate this phenomenon in a first example.

Example 1.1 ($n = 2, d = 5$)**.** Let Z be a set of 17 general points in the plane $\mathbb{P}^2$. The lowest degree elements of $I(Z)$ are in degree 5 and, a priori, $I(Z)$ can have minimal generators in degree 5 and 6. It turns out that $I(Z)$ is generated by four quintics. In particular, its chopped ideal $I(Z)_{\langle 5\rangle}$ coincides with $I(Z)$. We provide a simple snippet of code in `Macaulay2` [GS] to compute $I(Z)$, its chopped ideal, and the corresponding Hilbert functions: in this case, line 4 returns `true` and the two Hilbert functions coincide.

```
1 loadPackage "Points"
2 I = randomPoints(2,17);
3 Ichop = ideal super basis(5,I);
4 I == Ichop
5 for t to 10 list {t, hilbertFunction(t,I), hilbertFunction(t,Ichop)}
```

This uses the convenient package `Points.m2` to calculate the ideal `I` [SSS+]. Now let Z be a set of 18 general points in $\mathbb{P}^2$. In this case $I(Z)$ is generated by three quintics *and one sextic*. The chopped ideal $I_{\langle 5\rangle}$ is the ideal generated by the three quintics. Changing `17` into `18` in line 2 provides the code to compute the chopped ideal of $I(Z)$. Now, line 8 returns `false` and the two Hilbert functions are recorded below:

t	0	1	2	3	4	5	6	7	8	9	...
$h_{S/I(Z)}(t)$	1	3	6	10	15	18	18	18	18	18	...
$h_{S/I(Z)_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	**19**	18	18	18	...

We observe that the Hilbert polynomials are the same: they are equal to the constant 18. However, the Hilbert function of $S/I_{\langle 5\rangle}$ overshoots the Hilbert polynomial in degree 6, and then falls back to 18 in degree 7. This specific example is explained in detail in Section 3.1. More generally, the goal of this work is to better understand this phenomenon for all r. $\diamond$

We will see that the problem of understanding the Hilbert function of the chopped ideal of a set of points is related to several classical conjectures in commutative algebra and algebraic geometry, such as the Ideal Generation Conjecture and the Minimal Resolution Conjecture. Besides this, our motivation comes from computational geometry. In the most general setting, one is given a system of homogeneous polynomials with the task of determining the finite set of solutions Z. In a number of applications, the given polynomials generate a subideal of $I(Z)$. Often, this subideal is the chopped ideal $I_{\langle d\rangle}$. This happens, for instance, in classical tensor decomposition algorithms, see Section 6. In order to solve the resulting polynomial system using normal form methods, such as Gröbner bases, one constructs a *Macaulay matrix* of size roughly $h_S(d+e)$, where e is a positive integer such that $h_{S/I_{\langle d\rangle}}(d+e) = r$; see [EM99, Tel20] for details. Hence, it is important to answer the following question:

$$\text{“What is the smallest } e_0 > 0 \text{ such that } h_{S/I_{\langle d\rangle}}(d+e_0) = r\,?\text{”} \tag{2}$$

Example 1.1 analyzes two cases for $d = 5$ in $\mathbb{P}^2$: if $r = 17$ then the answer is $e_0 = 1$, and for $r = 18$ the answer is $e_0 = 2$. Interestingly, this means that finding 18 points from their vanishing quintics using normal form methods is significantly harder than finding 17 points.

The following conjecture predicts the Hilbert function of the chopped ideal $I_{\langle d\rangle}$.

Conjecture 1 (Expected Syzygy Conjecture)**.** *Let Z be a general set of r points in $\mathbb{P}^n$ and let d be the smallest value for which $h_S(d) \geq r$. Then for any $e \geq 0$*

$$h_{I_{\langle d\rangle}}(d+e) = \begin{cases} \displaystyle\sum_{k\geq 1} (-1)^{k+1} \cdot h_S(d+e-kd) \cdot \binom{h_S(d)-r}{k} & e < e_0 \\ h_S(d+e) - r & e \geq e_0 \end{cases} \tag{3}$$

where $e_0 > 0$ is the smallest integer such that the summation is at least $h_S(d+e_0) - r$.

The heuristic motivation for this conjecture is that, generically, the equations of degree d of a set of points are *as independent as possible*. More precisely, their syzygy modules are generated by the Koszul syzygies, as long as the upper bound $h_{I_{\langle d\rangle}} \leq h_{I(Z)}$ allows for it.

Our contribution is a proof of Conjecture 1 for many small values of n, r, and in several infinite families of pairs (n, r).

Theorem 1.2. *Conjecture 1 is true in the following cases:*

- *Theorem 4.1: $r_{d,\max} = h_S(d) - (n+1)$ for all d in all dimensions n;*
- *Theorem 3.5: $r_{d,\min} = \frac{1}{2}(d+1)^2$ when d is odd, in the case $n = 2$;*
- *Lemma 2.7: $r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ and $n \leq 4$ and more generally whenever the Ideal Generation Conjecture holds;*
- *Theorem 5.1: In a large number of individual cases in low dimension:*

n	2	3	4	5	6	7	8	9	10
r	≤ 2343	≤ 2296	≤ 1815	≤ 1272	≤ 908	≤ 767	≤ 479	≤ 207	≤ 267

We discuss the role of the Ideal Generation Conjecture mentioned in Theorem 1.2 in Section 2. We propose a second conjecture, implied by Conjecture 1, which pertains to question (2). For $Z \in (\mathbb{P}^n)^r$, let $I_{\langle d\rangle} = \langle I(Z)_d\rangle_S$ and define $\gamma_n(d, Z) = \min\{\, e \in \mathbb{Z}_{>0} \mid h_{S/I_{\langle d\rangle}}(d+e) = r \,\}$.

Conjecture 2 (Saturation Gap Conjecture)**.** *Let $n, d, r \in \mathbb{Z}_{>0}$ be integers with $r < h_S(d) - n$. For general Z, the value $\gamma_n(d, Z) = \gamma_n(d, r)$ only depends on n, d, r and it is given explicitly by*

$$\gamma_n(d, r) = \min\left\{ e \in \mathbb{Z}_{>0} \;\middle|\; h_S(d+e) - r \leq \sum_{k=1}^{n-3} (-1)^{k+1} h_S(d+e-kd) \binom{h_S(d)-r}{k} \right\}.$$

The fact that the *general* value $\gamma_n(d,r)$ in Conjecture 2 only depends on n, d, r is a consequence of a standard semicontinuity argument, see Proposition 5.2. We call this quantity the *saturation gap*. It measures the gap between degrees at which the chopped ideal $I_{\langle d\rangle}$ agrees with its saturation $(I_{\langle d\rangle})^{\mathrm{sat}} = I(Z)$. Theorem 1.2 guarantees that Conjecture 2 holds in all listed cases. Moreover, Corollary 4.5 provides an upper bound for $\gamma_n(d,r)$ for every n, d, r.

The paper is organized as follows. Section 2 sets the stage for the study of chopped ideals. It proves some preliminary results and explains the relations to other classical conjectures in commutative algebra and algebraic geometry. Section 3 is devoted to the case of points in the projective plane. It includes a detailed explanation of the case of 18 points in $\mathbb{P}^2$, the first non-trivial case, and two results solving Conjecture 1 in extremal cases. Section 4 concerns the proof of Conjecture 1 for the largest possible number of points $r = h_S(d)-(n+1)$ for given n, d. Moreover, we provide an upper bound for the saturation gap for any number of points. Section 5 contains a computational proof for the remaining cases in Theorem 1.2. Finally, Section 6 discusses symmetric tensor decomposition and its relation to Conjecture 2. The computations in the final two sections use `Macaulay2` [GS] and `Julia`; the code to replicate the computations is available online at https://mathrepo.mis.mpg.de/ChoppedIdeals/.

2 Chopped ideals

Definition 2.1 (Chopped ideal)**.** Let $I \subseteq S$ be a homogeneous ideal and $d \geq 0$. The *chopped ideal* in degree d associated to I is $I_{\langle d\rangle} \coloneqq \langle I_d \rangle_S$.

2.1 The chopping map

Denote by $U_{\mathrm{genHF}} \subseteq (\mathbb{P}^n)^r$ the dense Zariski open subset of $(\mathbb{P}^n)^r$ consisting of r-tuples satisfying (1). We focus on the chopped ideals $I_{\langle d\rangle}$ where $I = I(Z)$ for some $Z \in U_{\mathrm{genHF}}$. Moreover, we are interested in the case where Z can be computed from its chopped ideal. To this end, we determine the values of r for which $I(Z)$ and $I_{\langle d\rangle}$ define the same subscheme of $\mathbb{P}^n$. Given a set of homogeneous polynomials $J \subseteq S$, let $\mathrm{V}(J) \subseteq \mathbb{P}^n$ denote the subscheme of $\mathbb{P}^n$ that they define.

Theorem 2.2. *Let Z be a general set of r points, let $d \geq 1$ and $I_{\langle d\rangle} = I(Z)_{\langle d\rangle}$.*

(i) *If $r = h_S(d) - n$, then $\mathrm{V}(I_{\langle d\rangle})$ is a set of d^n reduced points.*

(ii) *If $r \geq h_S(d) - n$, then $\mathrm{V}(I_{\langle d\rangle})$ is a complete intersection of dimension $r + n - h_S(d)$.*

(iii) *If $r < h_S(d) - n$, then $\mathrm{V}(I_{\langle d\rangle})$ is the reduced scheme Z.*

For the proof, we consider a geometric interpretation of the operation of *chopping* an ideal.

Definition 2.3 (Chopping map $\mathfrak{c}$)**.** For given d with $h_S(d) \geq r$, the *chopping map* is

$$\mathfrak{c}\colon U_{\mathrm{genHF}} \to \mathrm{Gr}(h_S(d) - r, S_d), \quad Z \mapsto [I(Z)_d].$$

The chopping map is a morphism of varieties. In fact, there is a commutative diagram involving the Veronese embedding $\nu_d\colon \mathbb{P}^n \hookrightarrow \mathbb{P}(S_d^\vee)$

$$\begin{tikzcd}
\mathbb{P}(S_d^\vee)^r \supseteq (V_{d,n})^r \arrow[r, dashed, "\text{span}"] & \mathrm{Gr}(r, S_d^\vee) \arrow[d, "\cong"] \\
(\mathbb{P}^n)^r \supseteq U_{\text{genHF}} \arrow[u, hook, "(\nu_d)^{\times r}"] \arrow[r, "\mathfrak{c}"] & \mathrm{Gr}(h_S(d) - r, S_d).
\end{tikzcd}$$

For a linear space $T \subseteq S_d$, the scheme $\mathrm{V}(T) \subseteq \mathbb{P}^n$ is the intersection $\nu_d(\mathbb{P}^n) \cap \mathbb{P}(T^\perp) \subseteq \mathbb{P}(S_d^\vee)$ under the identification induced by the Veronese embedding, see, e.g. [Lan12, Prop. 4.4.1.1]. Notice that $\mathfrak{c}$ is invariant under permutation of the factors of $(\mathbb{P}^n)^r$, therefore it induces a map on the quotient $\widetilde{\mathfrak{c}}\colon U_{\text{genHF}}/\mathfrak{S}_r \to \mathrm{Gr}(h_S(d) - r, S_d)$.

Theorem 2.4 (Geometry of the chopping map)**.** *Let r, n, d be positive integers.*

(i) *If $r \geq h_S(d) - n$, then $\mathfrak{c}$ is dominant, with general fiber of dimension $nr - r(h_S(d) - r)$.*

(ii) *If $r \leq h_S(d) - n$, then $\mathfrak{c}$ is generically finite. More precisely, the induced map $\widetilde{\mathfrak{c}}$ has degree $\binom{d^n}{r}$ if $r = h_S(d) - n$, it is generically injective otherwise.*

Proof of Theorem 2.2 and Theorem 2.4. First consider the case $r = h_S(d) - n = 1 + \operatorname{codim} \nu_d(\mathbb{P}^n)$. A general linear space $\Lambda \in \mathrm{Gr}(r, S_d^\vee)$ intersects $\nu_d(\mathbb{P}^n)$ in a non-degenerate set of reduced points [Har92, Prop. 18.10]. Picking r points on $\nu_d(\mathbb{P}^n)$ spanning Λ, we see that the map $\mathfrak{c}$ is dominant. Furthermore, $\Lambda \cap \nu_d(\mathbb{P}^n)$ consists of $\deg V_{d,n} = d^n$ reduced points. By genericity, any subset of r points span Λ, hence $\widetilde{\mathfrak{c}}^{-1}(\Lambda^\perp)$ consists of $\binom{d^n}{r}$ points in $U_{\text{genHF}}/\mathfrak{S}_r$.

Next, let $r > h_S(d) - n$, and let $U' \subseteq (\mathbb{P}^n)^r$ be the open set from the previous case. For any set of r points $Z \in U_{\text{genHF}}$ and containing a subset Z' belonging to U', we must have $\dim \mathrm{V}(I(Z)_d) = \#(Z \setminus Z')$. Otherwise, the additional equations from $I(Z')_d$ could not cut down the dimension to 0. Since S is graded Cohen-Macaulay, this implies that a basis of $I(Z)_d$ is a regular sequence [Mat87, Thm. 17.4]. This shows that for any such Z, $\mathrm{V}(I(Z)_d)$ is a complete intersection of dimension $r - (h_S(d) - n)$. Proving that the chopping map is dominant is done exactly as in the previous case, the claim about the fiber dimension is a dimension count.

Finally, consider $r < h_S(d) - n$. We give a proof valid in characteristic 0. The classical *Multisecant Lemma* [Rus16, Prop. 1.4.3] states that a general k-secant plane to a non-degenerate projective variety $X \subseteq \mathbb{P}^N$ is not a $k+1$-secant for $k < \operatorname{codim} X$. Applying this to the Veronese variety, for a general set of $r < h_S(d) - n$ points Z, we have

$$\nu_d(\mathrm{V}(I(Z)_d)) = \nu_d(\mathbb{P}^n) \cap \mathbb{P}(I(Z)_d^\perp) = \nu_d(\mathbb{P}^n) \cap \langle \nu_d(Z) \rangle_{\mathbb{P}} = \nu_d(Z).$$

This shows that generically $\mathrm{V}(I(Z)_d) = Z$. In particular, $\widetilde{\mathfrak{c}}$ is generically injective. $\square$

This answers our question on when Z can be recovered from the chopped ideal.

Corollary 2.5. *With notation as before, for general Z the following are equivalent:*

(i) $\mathrm{V}(I(Z)_d) = Z$, *as reduced schemes;*

(ii) $(I_{\langle d\rangle})^{\mathrm{sat}} = I(Z)$, *where* $\cdot^{\mathrm{sat}}$ *denotes saturation with respect to* $\langle x_0, \ldots, x_n\rangle$*;*

(iii) $r < h_S(d) - n$ *or* $r = 1$ *or* $(n, r) = (2, 4)$.

Proof. By the projective Nullstellensatz, we have $J^{\mathrm{sat}} = I(\mathrm{V}(J))$, this shows the equivalence of (i) and (ii). If $r < h_S(d) - n$, then by Theorem 2.2 $\mathrm{V}(I(Z)_{\langle d\rangle}) = Z$. If $r \geq h_S(d) - n$, then $\mathrm{V}(I(Z)_{\langle d\rangle})$ is a complete intersection, which, for general Z, only happens if Z is a single point or four points in $\mathbb{P}^2$. $\square$

Inspecting the proof of Theorem 2.4, we make a useful technical observation.

Remark 2.6. For d such that $h_S(d) > r$ and Z general, a general collection of polynomials $f_1, \ldots, f_s \in I(Z)_d$ is a regular sequence, where $s = \min\{n, h_S(d) - r\}$.

2.2 The saturation gap and expected syzygies

From now on, for fixed n, r, set $d = \min\{\, t \mid h_S(t) \geq r \,\}$. Let Z be a set of r general points in $\mathbb{P}^n$, with vanishing ideal $I = I(Z)$. The degree d is the *Hilbert regularity* of Z (or $S/I(Z)$) defined for a finite graded S-module M by

$$\mathrm{reg}_{\mathrm{H}}(M) \coloneqq \min\{\, d \in \mathbb{Z} \mid h_M(t) = \mathrm{HP}_M(t) \text{ for } t \geq d \,\}.$$

The minimal generators of $I = I(Z)$ are in degrees $\{d, d+1\}$ [IK99, Thm. 1.69] and the operation of chopping the ideal in degree d is trivial unless I has generators in degree $d+1$. The number of minimal generators in degree d is $h_I(d) = h_S(d) - r$ by assumption (1), while the minimal generators in degree $d+1$ span a complement of $S_1 I_d$ in I_{d+1}. The linear space $S_1 I_d$ is the image of the multiplication map $\mu_1 \colon S_1 \otimes_{\Bbbk} I_d \to I_{d+1}$; its expected dimension is

$$\max\{h_S(1) \cdot h_I(d),\, h_I(d+1)\} = \max\{(n+1) \cdot (h_S(d) - r),\, h_S(d+1) - r\}.$$

which is always an upper bound and is achieved if and only if μ_1 has maximal rank. This leads to the following long standing conjecture [GM84].

Conjecture 3 (Ideal generation conjecture)**.** *Let* n, r, d *be as above. There is a Zariski dense open subset* $U_{\mathrm{igc}} \subseteq (\mathbb{P}^n)^r$ *such that, for* $Z \in U_{\mathrm{igc}}$*, the number of minimal generators of* $I(Z)$ *in degree* $d+1$ *is* $\max\{0, h_S(d+1) - r - (n+1) \cdot (h_S(d) - r)\}$.

Therefore, we see that $I(Z)$ has generators in degree $d+1$ if $h_S(d+1) - r - (h_S(d) - r)(n+1) > 0$, or equivalently

$$r > \frac{(n+1)h_S(d) - h_S(d+1)}{n}.$$

This bound is sharp assuming the ideal generation conjecture, which is known to hold for $n \leq 4$ or r large, see Section 2.3. In fact, using Corollary 2.5, we can pinpoint the range in which the chopped ideal cuts out Z in a non-saturated way.

Lemma 2.7. *Let n, d be positive integers. If*

$$\frac{(n+1)h_S(d) - h_S(d+1)}{n} \;<\; r \;<\; h_S(d) - n, \tag{4}$$

then a general set of r points in $\mathbb{P}^n$ has Hilbert regularity d, $\mathrm{V}(I(Z)_d) = Z$ but $I_{\langle d\rangle} \subsetneq I$. If the Ideal Generation Conjecture holds for n, d, then the lower bound is tight.

Remark 2.8. Note that $\frac{(n+1)h_S(d)-h_S(d+1)}{n} \geq h_S(d-1)$. In particular, in the interesting range, equations of degree d are equations of minimal degree.

In light of the Ideal Generation Conjecture, our Conjecture 1 is a natural generalization; it claims that the multiplication map $\mu_e \colon I_d \otimes_{\Bbbk} S_e \to I_{d+e}$ has the largest possible rank. To give a formal upper bound on the rank of μ_e, we introduce the lexicographic ordering on functions $h, h' \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$:

$$h \leq_{\text{lex}} h' \quad \text{if and only if} \quad \inf\{\, t \mid h(t) < h'(t)\,\} \leq \inf\{\, t \mid h(t) > h'(t)\,\}.$$

An important theorem of Fröberg [Frö85] asserts that for any ideal $J \subseteq S$ of depth 0 generated by $s \geq n+1$ elements in degree d one has

$$h_{S/J}(t) \;\geq_{\text{lex}}\; \text{frö}_{d,s}(t) := \begin{cases} \sum_{k\geq 0}(-1)^k h_S(t-kd)\binom{s}{k} & \text{if } t < t_0 \\ 0 & \text{if } t \geq t_0, \end{cases}$$

where $t_0 \geq 0$ is the first value for which the summation becomes nonpositive. The *Fröberg Conjecture* predicts that equality is achieved for general J. If the conjecture holds (for particular d, s) the lex-inequality is "upgraded" to a pointwise inequality for *all* J. In our situation this leads to the following theorem.

Theorem 2.9. *If $I \subseteq S$ has dimension 1, degree $r < h_S(d) - n$ and Hilbert function (1), then*

$$h_{S/I_{\langle d\rangle}}(t) \;\geq_{\text{lex}}\; \begin{cases} \text{frö}_{d,h_I(d)}(t) & \text{if } t < t_1 \\ r & \text{if } t \geq t_1, \end{cases}$$

where $t_1 = \inf\{\, t > d \mid \text{frö}_{d,h_I(d)}(t) \leq r\,\}$. More precisely, if $h_{S/I_{\langle d\rangle}}(t') \leq r$ for some $t' > d$, then $h_{S/I_{\langle d\rangle}}(t) = r$ for $t \geq t'$.

Proof. By [Frö85, Lem. 1], $h_{S/I_{\langle d\rangle}}(t) \geq h_{S/J}(t)$ where J is generated by $h_I(d)$ general forms of degree d. Applying Fröberg's Theorem from above to J, which has dimension and depth 0, we obtain $h_{S/I_{\langle d\rangle}} \geq_{\text{lex}} \text{frö}_{d,h_I(d)}$. Furthermore, if $h_{S/I_{\langle d\rangle}}(t') \leq r$ for some $t' > d$, then $(I_{\langle d\rangle})_{t'} = I_{t'}$. Since the minimal generators of $I(Z)$ are in degree at most $d+1 \leq t'$, we have $(I_{\langle d\rangle})_t = I_t$ for $t \geq t'$ and hence $h_{S/I_{\langle d\rangle}}$ sticks to r from that point on. $\square$

Conjecture 1 predicts that for Z general, the Hilbert function $h_{S/I(Z)_{\langle d\rangle}}$ satisfies Theorem 2.9 with equality, which then is upgraded to a pointwise lower bound. In particular, the multiplication map $\mu_e : I_d \otimes_{\Bbbk} S_e \to I_{d+e}$ is either surjective onto I_{d+e}, or it achieves the maximum possible dimension from Theorem 2.9:

$$h_{I_{\langle d\rangle}}(d+e) \;=\; \sum_{k\geq 1}(-1)^{k+1} \cdot h_S(d+e-kd) \cdot \binom{h_S(d)-r}{k}$$

until this sum falls below $h_I(d+e)$, from which point on $h_{I_{\langle d\rangle}}(d+e) = h_S(d+e) - r$.

2.3 Related open problems in commutative algebra

In this section, we give an overview of several conjectures in the study of ideals of points, related to Conjecture 1 and Conjecture 2. Let Z be a set of r general points in $\mathbb{P}^n$, let $I = I(Z)$ be the vanishing ideal and let d be the Hilbert regularity of Z.

The multiplication map $I_t \otimes S_e \to I_{t+e}$ is surjective for $t \geq d+1$ because all generators of $I(Z)$ are in degrees d and $d+1$. The already mentioned *Ideal Generation Conjecture* (IGC) stated in Conjecture 3, predicts that $\mu_1 \colon I_d \otimes S_1 \to I_{d+1}$ has full rank: in other words, either μ_1 is surjective or I has exactly $h_S(d+1) - (n+1)(h_S(d) - r)$ generators of degree $d+1$. This is related to Conjecture 1, which predicts that $\mu_e \colon I_d \otimes S_e \to S_{d+e}$ has the *expected rank*, and its kernel arises from the Koszul syzygies of the degree d generators of I.

The *Minimal Resolution Conjecture* (MRC) [Lor93] is a generalization of the IGC which predicts the entire *Betti table* of the ideal I. Consider the minimal free resolution of S/I, regarded as an S-module:

$$0 \longleftarrow S/I \longleftarrow S \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_{\mathrm{pd}(M)} \longleftarrow 0, \qquad F_i = \bigoplus_j S[-j]^{\beta_{i,j}}.$$

A consequence of [IK99, Lem. 1.69] is that, for $i \geq 1$, there are at most two nonzero Betti numbers; they are $\beta_{i,d+i-1}$ and $\beta_{i,d+i}$. The IGC predicts that either $\beta_{1,d+1} = 0$ or $\beta_{2,d+1} = 0$. The MRC predicts all values $\beta_{i,j}$.

Notice that if $\beta_{2,d+1} = 0$, then $\beta_{1+i,d+i} = 0$ for every $i \geq 1$ as well; in this case the values $\beta_{1,d}, \beta_{1,d+1}$, together with the exactness of the resolution, uniquely determine the other $\beta_{i,j}$. This is expected to be the case in the range of (4). In particular, in this range the MRC and the IGC are equivalent and, in a sense, Conjecture 1 is a generalization of both.

The MRC is known to be true for $n = 2$ [GM84, GGR86], for $n = 3$ [Bal87] and for $n = 4$ [Wal95]. Moreover, it has been proved in an asymptotic setting [HS96] and in a number of other sporadic cases, for which we refer to [EP96]. It is however false in general [EPSW02]. There are no known counterexamples to the IGC.

We record here the statement in the case of $\mathbb{P}^2$, where the Hilbert-Burch theorem dictates the structure of the minimal free resolution of S/I [Eis05, Thm. 3.2, Prop. 3.8].

Proposition 2.10 (Minimal resolution conjecture in $\mathbb{P}^2$)**.** *For a general collection of r points $Z \subseteq \mathbb{P}^2$ with $\mathrm{reg}_{\mathrm{H}}(Z) = d$, the minimal free resolution of $S/I(Z)$ has the form*

$$0 \longleftarrow S/I(Z) \longleftarrow S \longleftarrow \begin{matrix} S[-d]^{\beta_{1,d}} \\ \bigoplus \\ S[-(d+1)]^{\beta_{1,d+1}} \end{matrix} \xleftarrow{B} \begin{matrix} S[-(d+1)]^{\beta_{2,d+1}} \\ \bigoplus \\ S[-(d+2)]^{\beta_{2,d+2}} \end{matrix} \longleftarrow 0.$$

Here $\beta_{1,d} = h_S(d) - r$, $\beta_{1,d+1} = \max\{0, h_S(d+1) - (n+1)\beta_{1,d} - r\}$, $\beta_{2,d+1} + \beta_{2,d+2} = \beta_{1,d} + \beta_{1,d+1} - 1$ and $\beta_{1,d+1} \cdot \beta_{2,d+1} = 0$.

For a proof of this particular case, see for example [Sau85, Prop. 1.7]. In the paper the proof goes via polarization of monomial ideals. One might expect a similar approach would yield Conjecture 1 in $\mathbb{P}^2$. This is not the case, as we will show in Theorem 5.3.

The already mentioned Fröberg's Conjecture [Frö85] predicts the Hilbert function of the ideal generated by generic forms. Conjecture 1 states that chopped ideals of general points satisfy Fröberg's conjecture for as many values of $t \in \mathbb{Z}_{\geq 0}$ as they *possibly* can.

2.4 Castelnuovo-Mumford Regularity

We discussed relations of Conjecture 1 with the IGC and the MRC, which have a more cohomological flavour. This raises questions about other cohomological invariants of chopped ideals. We prove a statement regarding the *Castelnuovo-Mumford regularity* of $I_{\langle d \rangle}$. For an S-module M, this is defined as $\operatorname{reg}_{\mathrm{CM}} M = \max\{\, i+j \mid \beta_{i,j}(M) \neq 0 \,\}$, where $\beta_{i,j}$ are the graded Betti numbers of M.

Theorem 2.11. *Let $J \subseteq S$ be a one-dimensional graded ideal, then*

$$\operatorname{reg}_{\mathrm{CM}} S/J = \max\{\operatorname{reg}_{\mathrm{H}} S/J - 1,\, \operatorname{reg}_{\mathrm{H}} S/J^{\mathrm{sat}}\}. \tag{5}$$

Applying this theorem to a chopped ideal of a general set of points, we obtain:

Corollary 2.12. *Let n, r, d satisfy (4). Then for a general set of r points*

$$\operatorname{reg}_{\mathrm{CM}} S/I_{\langle d \rangle} = \operatorname{reg}_{\mathrm{H}} S/I_{\langle d \rangle} - 1.$$

The Conjecture 2 predicts the Hilbert regularity of $I_{\langle d \rangle}$, hence this conjecture is directly related to Castelnuovo-Mumford regularity.

Proof of Theorem 2.11. The proof relies on local cohomology. Let $\mathfrak{m} = \langle x_0, \dots, x_n \rangle_S$. The 0-th local cohomology group measures non-saturatedness as

$$\mathrm{H}^0_{\mathfrak{m}}(S/J) = \left\{\, x + J \in S/J \;\middle|\; \mathfrak{m}^k x \subseteq J,\ k \gg 0 \,\right\} = J^{\mathrm{sat}}/J.$$

The dimension of a finite S-module M can be characterized as the largest $i \geq 0$ with $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$ [Eis05, Prop. A1.16] so all cohomology groups $\mathrm{H}^i_{\mathfrak{m}}(S/J)$ vanish for $i \geq 2$.

We next provide a description of the remaining first local cohomology group. Let $I = J^{\mathrm{sat}}$ and let $Z = \operatorname{Proj} S/I \subseteq \mathbb{P}^n$ be the scheme defined by I. The quotients S/I and S/J have the same higher local cohomology. The comparison sequence for local and sheaf cohomology

$$0 \longrightarrow S/I \longrightarrow \bigoplus_{d \in \mathbb{Z}} \underbrace{\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_Z(d))}_{\cong \Bbbk^{\deg Z}} \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(S/I) \longrightarrow 0.$$

shows that $\mathrm{H}^1_{\mathfrak{m}}(S/I)_d$ is the cokernel of $(S/I)_d \hookrightarrow \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_Z(d))$. Introducing the notation $\operatorname{end}(N) := \sup\{\, t \in \mathbb{Z} \mid N_t \neq 0 \,\}$, this shows that $\operatorname{end}(\mathrm{H}^1_{\mathfrak{m}}(S/J)) + 1 = \operatorname{reg}_{\mathrm{H}} S/I =: d$.

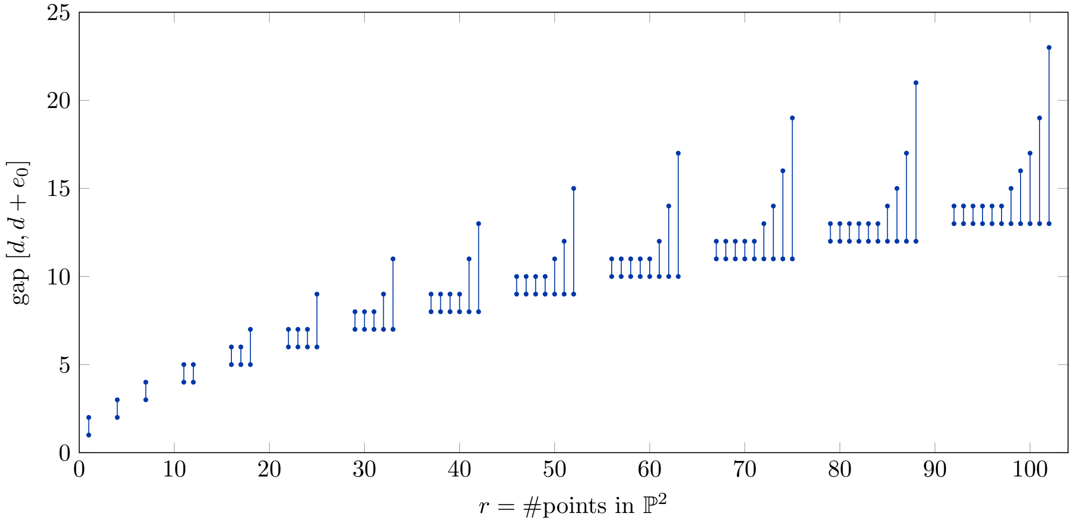

Figure 1: Saturation gaps for chopped ideals of points in the plane.

Now the Castelnuovo-Mumford regularity can be expressed in terms of local cohomology:

$$\operatorname{reg}_{\mathrm{CM}} M = \max_i \operatorname{end}(\mathrm{H}^i_{\mathfrak{m}}(M)) + i.$$

See [Eis05, Thm. 4.3]. For $M = S/J$ using vanishing in degree $i \geq \dim S/J = 1$, this gives

$$\operatorname{reg}_{\mathrm{CM}} S/J = \max\{\operatorname{end}(I/J),\, d\}. \tag{6}$$

To relate this to the maximum in equation (5), we distinguish two cases. If $\operatorname{end}(I/J) \geq d$, then $h_{S/J}(t) > h_{S/I}(t)$ for some $t \geq d$, so $\operatorname{end}(I/J) = \operatorname{reg}_{\mathrm{H}} S/J - 1$. Otherwise $\operatorname{end}(I/J) + 1 \leq d$, then $\operatorname{reg}_{\mathrm{H}} S/J \leq \operatorname{reg}_{\mathrm{H}} S/I$ and the maximum in (6) is attained at d. □

3 Points in the plane

When $n = 1$, Z is a set of points on the projective line. In this case $I(Z)$ is a principal ideal and it always coincides with its chopped ideal. In particular, Conjecture 1 and Conjecture 2 hold trivially, as well as the IGC and the MRC.

This section studies the case $n = 2$, that is when $Z \in (\mathbb{P}^2)^r$ is a collection of r general points in the plane. Figure 1 shows the saturation gaps for some values of r. In each case we use the chopped ideal $I_{\langle d \rangle}$ in degree $d = \min\{\, t \mid h_S(t) \geq r \,\}$. The gap is only plotted in cases where $I_{\langle d \rangle}$ defines Z scheme-theoretically, following Corollary 2.5. Since the IGC is known to be true in $\mathbb{P}^2$, Lemma 2.7 provides exactly the range where $I_{\langle d \rangle} \neq I(Z)$, but they both define the set Z as a scheme:

$$\frac{d(d+2)}{2} < r < \frac{(d+2)(d+1)}{2} - 2. \tag{7}$$

If $d < 5$, this range is empty and the corresponding gaps in Figure 1 have length 1. For $d = 5$, the only integer solution to (7) is $r = 18$. This is the leftmost length-two gap in Figure 1.

Hence, the simplest interesting chopped ideal is that of *three quintics passing through 18 general points in the plane*, first encountered in Example 1.1. In Section 3.1, we thoroughly work out this instructive example. For $d \geq 5$, write

$$r_{d,\min} = \left\lfloor \frac{d^2+2d+2}{2} \right\rfloor \quad \text{and} \quad r_{d,\max} = \frac{(d+2)(d+1)}{2} - 3 \tag{8}$$

for the extremal values in the range (7). Section 3.2 proves Conjecture 1 for $r = r_{d,\max}$, and Section 3.3 proves it for $r = r_{d,\min}$, when d is odd. Throughout this section, $S = \Bbbk[x_0, x_1, x_2]$.

3.1 Quintics through eighteen points

Let $Z = (z_1, \ldots, z_{18}) \in (\mathbb{P}^2)^{18}$ be a configuration of 18 general points in $\mathbb{P}^2$. Equation (1) guarantees that Z has no equations of degree 4 and exactly $3 = 21 - 18$ equations of degree 5. Hence $I(Z)_5 = \langle f_0, f_1, f_2 \rangle_\Bbbk$ for three linearly independent elements $f_i \in S_5$, and the chopped ideal is $I_{\langle 5 \rangle} = \langle f_0, f_1, f_2 \rangle_S$.

Notice that $h_{S/I_{\langle 5 \rangle}}(6) \geq 28 - 3 \cdot 3 = 19$ and equality holds if and only if the three quintics f_0, f_1, f_2 do not have linear syzygies. Since the IGC is true for $n = 2$, this is indeed the case. Moreover, Example 1.1 shows experimentally that $h_{S/I_{\langle 5 \rangle}}(7) = 18$, so the saturation gap is 2. The minimal resolution of $I(Z)$ according to Proposition 2.10 is

$$0 \longleftarrow I(Z) \longleftarrow \begin{matrix} S[-5]^3 \\ \bigoplus \\ S[-6] \end{matrix} \xleftarrow{\;B\;} S[-7]^3 \longleftarrow 0.$$

The minimal generators of $I(Z)$ are the maximal minors of the Hilbert-Burch matrix B, which is a 4×3-matrix with three rows of quadrics and one row of linear forms. As a result, the maximal minors are three quintics, spanning the linear space $I(Z)_5$, and one sextic, which is an element of $I(Z)_6 \setminus (I_{\langle 5 \rangle})_6$. The existence of the sextic is predicted by Theorem 2.9 and the gap $\gamma_n(d, r) = \gamma_2(5, 18) = 2$ agrees with Conjecture 2.

Note that the *missing sextic* is uniquely determined modulo the 9-dimensional linear space $(I_{\langle 5 \rangle})_6 \subseteq I(Z)_6$. We provide a way to compute an element of $I(Z)_6 \setminus (I_{\langle 5 \rangle})_6$ from Z.

Proposition 3.1. *Let $Z \subseteq \mathbb{P}^2$ be a set of 18 general points and let $Z = Z_1 \dot\cup Z_2$ be a partition of Z into two sets of 9 points. Let $f_1, f_2, f_3 \in I(Z)_5 \subseteq S_5$ be linearly independent and let $g_i \in I(Z_i)_3 \setminus \{0\} \subseteq S_3$. Then $g = g_1 g_2 \in I(Z)$ and $g \notin I_{\langle 5 \rangle}$.*

The proof of Proposition 3.1 is deferred to Appendix A.

It was observed in Example 1.1 that $(I_{\langle 5 \rangle})_7 = I(Z)_7$. This is equivalent to the following result, which is a consequence of the more general Theorem 3.3 and Theorem 3.5.

Proposition 3.2. *Let $Z \subseteq \mathbb{P}^2$ be a set of 18 general points and let $f_0, f_1, f_2 \in I(Z)_5 \subseteq S_5$ be linearly independent. Then f_0, f_1, f_2 have no quadratic syzygies. That is, $h_{S/I_{\langle 5 \rangle}}(7) = 18$.*

We sketch two different proofs of Proposition 3.2, to illustrate the idea of the more general proofs of Theorem 3.3, Theorem 3.5 and Theorem 4.1. A straightforward dimension count shows that $h_{S/I_{\langle 5\rangle}}(7) = 18$ if and only if the forms f_0, f_1, f_2 do not have quadratic syzygies. The Hilbert-Burch matrix B has the form

$$B = \begin{pmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \\ \ell_0 & \ell_1 & \ell_2 \end{pmatrix} \in S^{4\times 3},$$

where $q_{ij} \in S_2$ are quadrics, and $\ell_i \in S_1$ are linear forms. The quadratic syzygies of f_0, f_1, f_2 are the $\Bbbk$-linear combinations of the columns of B whose last entry is zero. If such a non-trivial $\Bbbk$-linear combination exists, the linear forms ℓ_0, ℓ_1, ℓ_2 are linearly dependent. Hence, $\mathrm{V}(\ell_0, \ell_1, \ell_2) \subseteq \mathbb{P}^2$ is non-empty. The quintics f_0, f_1, f_2 are the 3×3 minors of B involving the last row, so that $\mathrm{V}(\ell_0, \ell_1, \ell_2) \subseteq \mathrm{V}(f_0, f_1, f_2)$, showing $\mathrm{V}(\ell_0, \ell_1, \ell_2)$ is one of the points in Z. The genericity of Z, together with a monodromy argument, leads to a contradiction. Hence ℓ_0, ℓ_1, ℓ_2 are linearly independent, and one concludes that f_0, f_1, f_2 do not have quadratic syzygies. An analogous argument will give the proof of Theorem 3.5.

Alternatively, one can prove Proposition 3.2 via a classical liaison argument. Suppose $s_0 f_0 + s_1 f_1 + s_2 f_2 = 0$ is a quadratic syzygy of f_0, f_1, f_2, for some $s_j \in S_2$. Let $K \subseteq \mathbb{P}^2$ be the set of points defined by the ideal $\langle f_1, f_2\rangle_S$. By Bézout's theorem, we see that K is a complete intersection of 25 reduced points, and Z is a subset of K. In other words, $K = Z \,\dot{\cup}\, Z'$ where Z' is a set of 7 points, called the *liaison of Z in K*. For every $z \in Z'$, we have $s_0(z) f_0(z) = 0$. We have $f_0(z) \neq 0$, otherwise $z \in Z$, and we conclude that $s_0 \in I(Z')$. On the other hand, the theory of liaison guarantees that Z' has no nonzero quadratic equations. We conclude that $s_0 = 0$, and analogously for $s_1 = s_2 = 0$. This argument generalizes to a proof of Theorem 3.3 and Theorem 4.1.

3.2 The case $r = r_{d,\max}$

In this section, we prove Conjecture 1 and Conjecture 2 for the maximal number of points $r = r_{d,\max}$ in the plane. Fix $d \geq 5$ and let $Z \in (\mathbb{P}^2)^{r_{d,\max}}$ be a general collection of $r_{d,\max}$ points. By construction, $I(Z)$ is zero in degree smaller than d and the chopped ideal $I(Z)_{\langle d\rangle}$ has 3 generators of degree d. By Corollary 2.5, $I(Z)_{\langle d\rangle}$ defines Z scheme-theoretically. Since the IGC holds in $\mathbb{P}^2$, the three generators of degree d have no linear syzygies and $I(Z)$ has $d-4$ minimal generators of degree $d+1$. The minimal free resolution of $I(Z)$ is

$$0 \longleftarrow I(Z) \longleftarrow \begin{matrix} S[-d]^3 \\ \bigoplus \\ S[-(d+1)]^{d-4} \end{matrix} \xleftarrow{\;B\;} S[-(d+2)]^{d-2} \longleftarrow 0. \tag{9}$$

Conjecture 1 predicts the value for $h_{I_{\langle d\rangle}}$ in degree $d+e$:

$$h_{I(Z)_{\langle d\rangle}}(d+e) = \min\left\{3\cdot\binom{e+2}{2}, \binom{d+e+2}{2} - \binom{d+2}{2} + 3\right\}.$$

For $e = d-3$, both arguments give the minimum:

$$3 \cdot \binom{d-1}{2} = \frac{3d^2-9d+6}{2} = \binom{2d-1}{2} - \binom{d+2}{2} + 3.$$

Hence, we expect the map $\mu_{d-3}\colon S_{d-3} \otimes I(Z)_d \to I(Z)_{2d-3}$ to be an isomorphism.

Theorem 3.3. *Let $Z \in (\mathbb{P}^2)^{r_{d,\max}}$ be a collection of $r_{d,\max}$ general points, and let $I_{\langle d\rangle} = \langle I(Z)_d \rangle$ be its chopped ideal. The Hilbert function of $I_{\langle d\rangle}$ satisfies*

$$\begin{aligned} &h_{I_{\langle d\rangle}}(t) = 0 && \text{if } t < d, \\ &h_{I_{\langle d\rangle}}(t) = 3 \cdot h_S(t-d) && \text{if } d \le t \le 2d-3, \\ &h_{I_{\langle d\rangle}}(t) = h_S(t) - r_{d,\max} && \text{if } t \ge 2d-3. \end{aligned}$$

In this case, Conjecture 1 and Conjecture 2 hold and $\gamma_2(d, r_{d,\max}) = d-3$.

Proof. It suffices to show that the three generators f_0, f_1, f_2 of the chopped ideal do not have syzygies in degree $d-3$. This guarantees that the inequality in Theorem 2.9 is a pointwise upper bound, and in turn that equality holds. Suppose (s_0, s_1, s_2) is a syzygy of degree $d-3$:

$$s_0 f_0 + s_1 f_1 + s_2 f_2 = 0 \qquad \text{for some } s_i \in S_{d-3}.$$

We are going to prove that $s_0 = s_1 = s_2 = 0$. By Remark 2.6 we may assume that f_1, f_2 generate a complete intersection ideal defining a set K of d^2 distinct points in $\mathbb{P}^2$. The set K contains Z, and we write $Z' = K \setminus Z$ for the complement of Z in K.

It suffices to show that $I(Z')_{d-3} = 0$. Indeed, for every $z \in Z'$ we have $f_1(z) = f_2(z) = 0$, which implies $f_0(z)s_0(z) = 0$. But $f_0(z) \neq 0$ because by Corollary 2.5 the chopped ideal defines Z scheme-theoretically and $z \notin Z$. Hence $s_0 \in I(Z')_{d-3}$. If $I(Z')_{d-3} = 0$, we obtain $s_0 = 0$. This implies $s_1 = s_2 = 0$ as well, because s_1, s_2 defines a syzygy of the complete intersection $\langle f_1, f_2 \rangle_S$, which does not have non-trivial syzygies in degree smaller than d.

We are left with showing that $I(Z')_{d-3} = 0$. The Hilbert-Burch matrix B' of Z' can be obtained from the Hilbert-Burch matrix B of Z in (9) as follows; see, e.g., [Sau85, Prop. 1.3]. The entries of B in its first three rows are quadrics, and the ones in the remaining $d-4$ rows are linear forms. The second and third row of B correspond to f_1, f_2. The Hilbert-Burch matrix B' is the transpose of the submatrix obtained from B by removing the two rows corresponding to f_1, f_2. Therefore B' is a $(d-2) \times (d-3)$ matrix whose first column consists of quadratic forms, and the remaining $d-4$ columns consist of linear forms. The maximal minors of B' have degree $d-2$, and they are minimal generators of $I(Z')$ by the Hilbert-Burch Theorem. In particular, $I(Z')_{d-3} = 0$, as desired. □

Theorem 3.3 is a special version of Theorem 4.1, whose proof resorts to liaison theory in higher dimension and is less explicit. Therefore, we chose to include both proofs.

Theorem 3.3 allows us to provide an upper bound on the saturation gap for any set of general points in $\mathbb{P}^2$. Let $r \le r_{d,\max}$ and fix r general points Z in $\mathbb{P}^2$. By definition, $\gamma_2(d,r) = \min\{ e \in \mathbb{Z}_{>0} \mid h_{S/I_{\langle d\rangle}}(d+e) = r \}$, where $I_{\langle d\rangle} = \langle I(Z)_d \rangle_S$ for r general points Z.

Corollary 3.4. *For $r \le r_{d,\max}$ general points in the plane, the saturation gap $\gamma_2(d,r)$ is at most $d-3$. In particular, the alternating sum in Conjecture 1 reduces to a single summand.*

Corollary 3.4 is a special case of Corollary 4.5 below.

3.3 The case $r = r_{d,\min}$ for odd d

Let $d = 2\delta+1$ be odd, and let Z be a set of $r = r_{d,\min} = 2(\delta+1)^2$ general points in $\mathbb{P}^2$. By Proposition 2.10, $I(Z)$ is generated by $\delta+1$ forms of degree d and 1 form of degree $d+1$:

$$I(Z) = \langle f_0, \ldots, f_\delta, g\rangle_S,$$

with $f_0,\ldots,f_\delta \in S_d$ and $g \in S_{d+1}$. In this section we prove the following result.

Theorem 3.5. *Let $d = 2\delta+1$ and let $Z \in (\mathbb{P}^2)^{r_{d,\min}}$ be a collection of $r_{d,\min} = 2(\delta+1)^2$ general points. The Hilbert Function of $I_{\langle d\rangle} = \langle I(Z)_d\rangle_S$ satisfies*

$$\begin{aligned} h_{S/I_{\langle d\rangle}}(d) &= r_{d,\min}, \\ h_{S/I_{\langle d\rangle}}(d+1) &= r_{d,\min}+1, \\ h_{S/I_{\langle d\rangle}}(t) &= r_{d,\min} \qquad\qquad \textit{if } t \ge d+2. \end{aligned}$$

In this case, Conjecture 1 and Conjecture 2 hold and $\gamma_2(d, r_{d,\min}) = 2$.

The proof uses the minimal free resolution of $I(Z)$, obtained from Proposition 2.10

$$0 \longleftarrow I(Z) \longleftarrow \begin{array}{c} S[-d]^{\delta+1} \\ \bigoplus \\ S[-(d+1)] \end{array} \xleftarrow{\;B\;} S[-(d+2)]^{\delta+1} \longleftarrow 0.$$

The Hilbert-Burch matrix B has the following form:

$$B = \begin{pmatrix} q_{00} & \cdots & q_{0\delta} \\ \vdots & & \vdots \\ q_{\delta 0} & \cdots & q_{\delta\delta} \\ \ell_0 & \cdots & \ell_\delta \end{pmatrix},$$

for some quadratic forms $q_{ij} \in S_2$ and linear forms $\ell_j \in S_1$. The degree e syzygies of $f_0,\ldots,f_\delta$ are the elements of the $\Bbbk$-vector space

$$\mathrm{Syz}(f_0,\ldots,f_\delta)_e \;=\; \left\{\, (s_0,\ldots,s_\delta) \in (S_e)^{\delta+1} \;\middle|\; s_0 f_0 + \cdots + s_\delta f_\delta = 0 \,\right\}.$$

Proof of Theorem 3.5. The fact that $h_{S/I_{\langle d\rangle}}(d) = h_{S/I}(d) = r_{d,\min}$ follows by (1). The IGC holds for $n=2$, and it implies $h_{S/I_{\langle d\rangle}}(d+1) = r_{d,\min}+1$.

For the statement on $h_{S/I\langle d\rangle}(t)$ for $t \ge d+2$, observe that $(\delta+1)\cdot h_S(2) = h_S(d+2) - r_{d,\min} + (\delta-2)$. Hence, it suffices to show that the forms $f_0,\ldots,f_\delta$ generating the chopped ideal $I_{\langle d\rangle} = \langle I(Z)_d\rangle_S$ have exactly $\delta-2$ quadratic syzygies, i.e.

$$\dim_{\Bbbk} \mathrm{Syz}(f_0,\ldots,f_\delta)_2 \;=\; \delta-2.$$

It is clear that $\dim_{\Bbbk} \operatorname{Syz}(f_0, \ldots, f_\delta)_2 \geq h_I(d)h_S(2) - h_I(d+2) = \delta - 2$; this also follows from Theorem 2.9. We show that $f_0, \ldots, f_\delta$ cannot have $\delta - 1$ syzygies.

The linear span $L_Z = \langle \ell_0, \ldots, \ell_\delta \rangle_{\Bbbk}$ of the linear forms in the last row of B does not depend on the choice of the minimal free resolution. This is a consequence of [Eis95, Thm. 20.2].

If $f_0, \ldots, f_\delta$ have $\delta - 1$ quadratic syzygies for a general choice of Z, then $\dim_{\Bbbk} L_Z$ is at most 2, which implies that the variety $\mathrm{V}(L_Z)$ is non-empty. Moreover, by the Hilbert-Burch Theorem, the ideal generated by L_Z contains $I_{\langle d \rangle} = \langle f_0, \ldots, f_\delta \rangle_S$, which cuts out Z scheme-theoretically by Theorem 2.4. Hence $\dim_{\Bbbk} L_Z = 2$ and $\mathrm{V}(L_Z)$ is one of the points in Z. Define

$$\psi\colon (\mathbb{P}^2)^r \dashrightarrow \mathbb{P}^2, \qquad \psi(Z) \coloneqq \mathrm{V}(L_Z).$$

This is a rational map with the property that $\psi(Z) \in Z$. By definition, ψ is invariant under the action of the symmetric group $\mathfrak{S}_r$ permuting the factors of $(\mathbb{P}^2)^r$. Consider the subvarieties of $(\mathbb{P}^2)^r$ defined by

$$Y_j \coloneqq \overline{\{\, Z = (z_1, \ldots, z_r) \in \operatorname{dom}(\psi) \mid \psi(Z) = z_j \,\}}.$$

We have $(\mathbb{P}^2)^r = \bigcup_{j=1}^r Y_j$. Since $(\mathbb{P}^2)^r$ is irreducible, we have $Y_j = (\mathbb{P}^2)^r$ for at least one j. On the other hand, $\mathfrak{S}_r$ invariance implies that if $Y_j = (\mathbb{P}^2)^r$ for one j, then this must be true for all j's. But any two Y_j are distinct because generically Z consists of distinct elements. This gives a contradiction showing that the map ψ cannot exist. We obtain that $\dim L_Z = 3$ for general Z and this concludes the proof. ☐

4 The largest possible saturation gap

In this section, we consider a set Z of $r = \binom{d+n}{n} - (n+1)$ general points in $\mathbb{P}^n$. Corollary 2.5 guarantees this is the largest possible number of points such that the chopped ideal $I_{\langle d \rangle} = \langle I(Z)_d \rangle_S$ in the ring $S = \Bbbk[x_0, \ldots, x_n]$ defines Z scheme-theoretically. We will show that Conjecture 1 is true in this case:

Theorem 4.1. *Let n, d be positive integers and let $Z \subseteq \mathbb{P}^n$ be a set of $r = h_S(d) - (n+1)$ general points. The Hilbert function of the chopped ideal $I_{\langle d \rangle} = \langle I(Z)_d \rangle_S$ satisfies*

$$h_{S/I_{\langle d \rangle}}(d+e) = \sum_{k \geq 0} (-1)^k h_S(d + e - kd) \binom{n+1}{k},$$

for $e \leq (n-1)d - (n+1)$. Conjecture 1 and Conjecture 2 hold with $\gamma_n(d,r) = (n-1)d - (n+1)$.

As a consequence of Theorem 4.1, we will obtain an upper bound on the saturation gap $\gamma_n(d,r)$ for all $r < h_S(d) - n$ in Corollary 4.5.

The proof of Theorem 4.1 relies on a fundamental fact in the theory of liaison. Given the Hilbert function of a set of points Z, one can compute the Hilbert function of the complementary set of points $Z' = K \setminus Z$ in a complete intersection $K \supseteq Z$. We record this fact in Proposition 4.4 below. In order to state it precisely, we introduce the following notation. The function $\Delta h_Z(t) = h_Z(t) - h_Z(t-1)$, with $h_Z = h_{S/I(Z)}$, is the *first difference* of the Hilbert function of Z. Often, $\Delta h_Z(t)$ is called the h-vector of Z and it is recorded as the sequence of its non-zero values. We record some immediate properties, see e.g. [Chi19].

Lemma 4.2. *For a finite set of points $Z \subseteq \mathbb{P}^n$, we have*

- $h_Z(t) = \sum_{t'=0}^{t} \Delta h_Z(t')$;
- $\mathrm{reg}_{\mathrm{H}}(Z) = \max\{\, t \mid \Delta h_Z(t) > 0 \,\}$;
- *if* $I(Z)_t = 0$ *then* $\Delta h_Z(t') = \binom{t'+n-1}{n-1}$ *for* $t' \leq t$.

For complete intersections K, the function Δh_K has a symmetry property [Eis95, Ch. 17].

Lemma 4.3. *Let $K \subseteq \mathbb{P}^n$ be a set of $d_1 \cdots d_n$ points whose ideal is generated by a regular sequence $f_1, \ldots, f_n$, with $\deg(f_i) = d_i$. Then $\mathrm{reg}_{\mathrm{H}}(K) = d_1 + \cdots + d_n - n$. In particular, if $d_1 = \cdots = d_n = d$, $\mathrm{reg}_{\mathrm{H}}(K) = n(d-1)$. Moreover, Δh_K is symmetric, that is, setting $\rho = \mathrm{reg}_{\mathrm{H}}(K)$, we have $\Delta h_K(t) = \Delta h_K(\rho - t)$.*

The theory of liaison studies the relation between the distinct irreducible components (or union of such) of a complete intersection; we only illustrate one result in the context of ideals of points; we refer to [MN02, Mig98] for an extensive exposition of the subject. The following is a consequence of [Mig98, Prop. 5.2.1]; see also [AC22, Eqn. (3)].

Proposition 4.4. *Let $Z \subseteq \mathbb{P}^n$ be a set of points and let $f_1, \ldots, f_n \in I(Z)$ be homogeneous polynomials of degree $d_1, \ldots, d_n$ defining a smooth complete intersection K of degree $d_1 \cdots d_n$. Let $Z' = K \setminus Z$ be the complement of Z in K. Let $\rho = \mathrm{reg}_{\mathrm{H}}(K) = d_1 + \cdots + d_n - n$. Then $\Delta h_Z(t) + \Delta h_{Z'}(\rho - t) = \Delta h_K(t)$.*

In words, Proposition 4.4 says that the sequence $\Delta h_{Z'}$ equals the sequence $\Delta h_K - \Delta h_Z$, in the reversed order.

The proof of this result uses a construction known as the *mapping cone* in homological algebra. The key fact is that the resolution of $I(Z')$ can be obtained from that of $I(Z)$ using the fact that the resolution of $I(K)$ is the classical Koszul complex of $f_1, \ldots, f_n$ [Mig98].

We now have all the ingredients to give a proof of Theorem 4.1.

Proof of Theorem 4.1. By construction, we have $\dim_{\Bbbk}(I_{\langle d \rangle})_d = \dim_{\Bbbk} I(Z)_d = n+1$. Let $f_0, \ldots, f_n \in I(Z)_d$ be general, so that $I_{\langle d \rangle} = \langle f_0, \ldots, f_n \rangle_S$ and $f_1, \ldots, f_n$ define a reduced complete intersection $K \subseteq \mathbb{P}^n$ of d^n-many points by Remark 2.6.

The Hilbert function $h_K(t)$ has the following compact form, which can be computed directly from the dimension of the syzygy modules in the Koszul complex [Eis95, Ch. 17]:

$$h_K(t) = \sum_{k=0}^{n} (-1)^k h_S(t - kd) \binom{n}{k}.$$

In particular $h_K(t) = h_Z(t)$ for $t \leq d-1$, and $h_K(d) = h_Z(d) + 1$. Recall from Lemma 4.3 that $\mathrm{reg}_{\mathrm{H}}(K) = n(d-1)$. Set $\rho \coloneqq n(d-1)$ and let $Z' \coloneqq K \setminus Z$ be the complement of Z in

K. By Lemma 4.3 and Proposition 4.4, we have

$$\Delta h_Z(t) + \Delta h_{Z'}(\rho - t) = \Delta h_K(t) = \Delta h_K(\rho - t).$$

Since $\Delta h_Z(t) = 0$ for $t \geq d+1$, we deduce $\Delta h_{Z'}(t) = \Delta h_K(t)$ for $t \leq \rho - (d+1)$. Since $I(Z') \subseteq I(K)$, this implies $I(Z')_t = I(K)_t$ for $t \leq \rho - (d+1)$. In particular, we obtain that

$$\dim I(Z')_t = h_S(t) - h_K(t) = \sum_{k \geq 1} (-1)^{k+1} h_S(t - kd) \binom{n}{k},$$

for $t \leq (n-1)d - (n+1)$. See Figure 2 for a schematic representation.

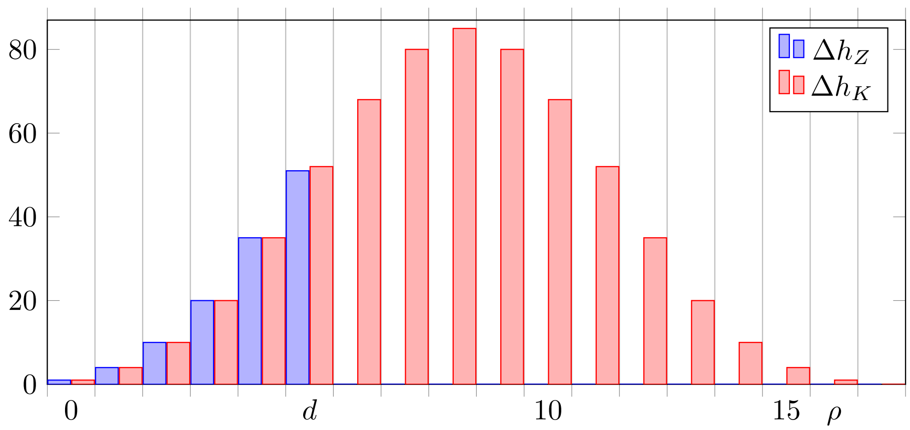

Figure 2: The sequences Δh_Z and Δh_K for $r = 121$ points in $\mathbb{P}^4$. Their difference, in reversed order, equals $\Delta h_{Z'}$. Note $\Delta h_K(5) = \Delta h_Z(5) + 1$.

Now, fix $e \leq nd - n - (d+1)$ and let $\mathrm{Syz}_e \subseteq S_e^{n+1}$ be the subspace of syzygies of degree e; i.e. the tuples $(s_0, \ldots, s_n)$ with $s_0 f_0 + \cdots + s_n f_n = 0$.

Consider the projection $\pi_0 \colon \mathrm{Syz}_e \to S_e$ onto the 0-th component S_e, that is $\pi(s_0, \ldots, s_n) = s_0$. We observe that the image of this map is contained in $I(Z')$. Indeed, since $Z' \subseteq K$, for every $p \in Z'$ we have $f_j(p) = 0$ for $j = 1, \ldots, n$. This implies $s_0(p) f_0(p) = 0$ for $p \in Z'$. Since $I_{\langle d \rangle}$ defines Z scheme-theoretically, we deduce that $f_0(p) \neq 0$ for every $p \in Z'$; hence $s_0(p) = 0$ for $p \in Z'$, as desired. Therefore, the image of the map π_0 is contained in $I(Z')_e$.

The kernel of π_0 consists of elements $\boldsymbol{s} = (0, s_1, \ldots, s_n) \in S_e^{n+1}$ such that $s_1 f_1 + \cdots + s_n f_n = 0$; hence $(s_1, \ldots, s_n)$ defines a syzygy of $f_1, \ldots, f_n$. Since the ideal $\langle f_1, \ldots, f_n \rangle_S$ is a complete intersection, its only syzygies in degree e are generated by the Koszul syzygies, and we deduce

$$\dim(\mathrm{Ker}\, \pi_0) = \sum_{k \geq 2} (-1)^k h_S(d + e - kd) \binom{n}{k}.$$

Since $\dim \mathrm{Syz}_e = \dim(\mathrm{Ker}\, \pi_0) + \dim \mathrm{Im}(\pi_0) \leq \dim(\mathrm{Ker}\, \pi_0) + \dim I(Z')_e$, the standard identity

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ yields

$$\begin{aligned}\dim \mathrm{Syz}_e \;&\leq\; \sum_{k\geq 1}(-1)^{k+1}h_S(e-kd)\binom{n}{k+1} + \sum_{k\geq 1}(-1)^{k+1}h_S(e-kd)\binom{n}{k}\\ &= \sum_{k\geq 1}(-1)^{k+1}h_S(e-kd)\binom{n+1}{k+1}.\end{aligned}$$

We conclude that for $e \leq (n-1)d-(n+1)$,

$$h_{I_{\langle d\rangle}}(d+e) \;=(n+1)\cdot h_S(e) - \dim \mathrm{Syz}_e \;\geq\; \sum_{k\geq 1}(-1)^{k+1}h_S(d+e-kd)\binom{n+1}{k}.$$

This shows that the ideal $I_{\langle d\rangle}$ satisfies Theorem 2.9 with equality, which then implies a point-wise equality.

It remains to show that $h_{S/I_{\langle d\rangle}}(d+e) \geq h_S(d+e)-(n+1)$ when $e \leq (n-1)d-(n+1)$ and equality holds only for $e=(n-1)d-(n+1)$. Set $e_0=(n-1)d-(n+1)$ and notice $d+e_0 = nd-n-1$. Let $M=(x_0^d,\ldots,x_n^d)$ which is a complete intersection ideal defining a 0-dimensional scheme supported at the origin in $\mathbb{A}^{n+1}$. Notice $h_{S/I_{\langle d\rangle}}(d+e)=h_{S/M}(d+e)$ for $e\leq e_0$. In particular, for $e=e_0$,

$$h_{S/I_{\langle d\rangle}}(d+e_0) = h_{S/M}((n+1)(d-1)-d);$$

this is the number of standard monomials of M of degree $(n+1)(d-1)-d$. These are the quotients of the form $(x_0^{d-1}\cdots x_n^{d-1})/\boldsymbol{x}^\alpha$ where $\boldsymbol{x}^\alpha$ is a monomial of degree d different from $x_0^d,\ldots,x_n^d$. Hence, we have $h_{S/I_{\langle d\rangle}}(d+e_0)=h_S(d)-(n+1)$, as desired.

If $e<e_0$, the same argument shows $h_{S/I_{\langle d\rangle}}(d+e) > h_S(d)-r$ because the value of the Hilbert function coincides with the number of divisors of $(x_0^{d-1}\cdots x_n^{d-1})$ of degree $(n+1)(d-1)-d-(e_0-e)$, which is greater than r. $\square$

This leads to the following generalization of Corollary 3.4.

Corollary 4.5. *For $r \leq h_S(d)-(n+1)$ general points in the plane, the saturation gap $\gamma_n(d,r)$ is at most $(n-1)d-(n+1)$. The sum in Conjecture 1 reduces to $n-1$ terms.*

Proof. The proof is by reverse induction on r. The base case is $r=h_S(d)-(n+1)$, for which the statement follows from Theorem 4.1. We are going to show that $\gamma_n(d,r-1)\leq\gamma_n(d,r)$. Let $Z_{r-1}=(z_1,\ldots,z_{r-1})\in(\mathbb{P}^n)^{r-1}$ be general, let z_r be one additional general point and set $Z_r=(z_1,\ldots,z_r)$. Let $e_0 \coloneqq \gamma_n(d,r)$, we have $h_{S/I(Z_r)_{\langle d\rangle}}(d+e_0)=r$ and need to show $h_{S/I(Z_{r-1})_{\langle d\rangle}}(d+e_0)=r-1$. For this is suffices to show

$$(I(Z_{r-1})_{\langle d\rangle})_{d+e_0} \;=\; S_{e_0}\cdot I(Z_{r-1})_d \supsetneq S_{e_0}\cdot I(Z_r)_d \;=\; (I(Z_r)_{\langle d\rangle})_{d+e_0}.$$

By genericity of Z_r, we can pick $f\in I(Z_{r-1})_d\setminus I(Z_r)_d$ and $h\in S_{e_0}$ not vanishing on z_r, then $fh\in S_{e_0}\cdot I(Z_{r-1})_d$, but $fh\notin I(Z_r)\supseteq I(Z_r)_{\langle d\rangle}$. $\square$

5 Proofs via computer algebra

This section provides a computational proof of Conjecture 1 for many small values of d, n, r.

Theorem 5.1. *Conjecture 1 holds for a set of r general points in $\mathbb{P}^n$, with the following values of n and r:*

n	2	3	4	5	6	7	8	9	10
r	≤ 2343	≤ 2296	≤ 1815	≤ 1272	≤ 908	≤ 767	≤ 479	≤ 207	≤ 267

The proof of Theorem 5.1 is computational. For every such (n, r) of interest, we exhibit an r-tuple $Z \in (\mathbb{P}^n_{\mathbb{Q}})^r$ for which the statement holds. The following semicontinuity result guarantees that this suffices to conclude.

Proposition 5.2. *Let $r < h_S(d) - n$ and let $U_{\text{genHF}} \subseteq (\mathbb{P}^n)^r$ be the collections of points satisfying (1). The set $U_{k,e} = \{ Z \in U_{\text{genHF}} \mid h_{I(Z)_{\langle d \rangle}}(d+e) \geq k \}$ is Zariski open in $(\mathbb{P}^n)^r$. In particular, the set $U = \{Z \in (\mathbb{P}^n)^r : h_{I_{\langle d \rangle}}$ satisfies (3)$\}$ is Zariski open.*

Proof. Let $\mathrm{Gr}(h_S(d) - r, S_d)$ be the Grassmannian of planes of dimension $h_S(d) - r$ in S_d. Consider the vector bundle $\mathcal{E} = \mathrm{Hom}(\mathcal{S} \otimes S_e, S_{d+e})$, where $\mathcal{S}$ denotes the tautological bundle over $\mathrm{Gr}(h_S(d) - r, S_d)$: the fiber of $\mathcal{E}$ at a plane $[L]$ is $\mathcal{E}_L = \mathrm{Hom}(L \otimes S_e, S_{d+e})$. The multiplication map $\mu : S_d \otimes S_e \to S_{d+e}$ defines a global section of $\mathcal{E}$ via restriction. The pull-back $\mathfrak{c}^*\mathcal{E}$ of $\mathcal{E}$ via the chopping map of Section 2 defines a bundle over U_{genHF} and $\mathfrak{c}^*\mu$ defines a global section of $\mathfrak{c}^*\mathcal{E}$. The set $U_{k,e}$ is the complement of the degeneracy locus

$$V_{k,e} = \left\{ Z \in U_{\text{genHF}} : \mathrm{rank}\big(\mu_e \colon I(Z)_d \otimes S_e \to S_{d+e}\big) < k \right\}.$$

This shows that $V_{k,e}$ is Zariski closed, hence $U_{k,e}$ is Zariski open.

The set U is the intersection of the finitely many open sets $U_{k_e,e}$ for $e = 1, \ldots, m_r$: here m_r is any upper bound on the saturation gap, for instance $m_r = (n-1)d - (n+1)$ from Corollary 4.5; k_e is the expected value of $h_{I_{\langle d \rangle}}(d+e)$ in (3). This concludes the proof. ☐

Theorem 5.1 is a direct consequence of Proposition 5.2.

Proof of Theorem 5.1. Identify the field of rational numbers $\mathbb{Q}$ with the prime field of $\Bbbk$. For every (n, r) of interest, we exhibit an instance $Z \in U_{\text{genHF}} \subseteq (\mathbb{P}^n(\mathbb{Q}))^r$ for which the Hilbert function of $\langle I(Z)_d \rangle_S$ satisfies (3). These instances can be found online at

https://mathrepo.mis.mpg.de/ChoppedIdeals/.

This guarantees that the corresponding open set U of Proposition 5.2 is non-empty, and therefore it is Zariski dense. This shows that for a general instance $Z \in (\mathbb{P}^n)^r$, the Hilbert function of Z satisfies (3), and therefore Conjecture 1 holds. ☐

Notice that it suffices to check (3) for e up to the expected saturation gap of Conjecture 2. Indeed, if it holds up to that value, we have $(I_{\langle d\rangle})_{d+e} = I(Z)_{d+e}$ for higher e, which is enough to conclude.

To speed up the computations, we make the following observation. Let $I \subseteq S = \Bbbk[x_0, \ldots, x_n]$ be an ideal generated by polynomials $f_1, \ldots, f_s$ with coefficients in $\mathbb{Z}$; here $\mathbb{Z} \subseteq \Bbbk$ is naturally identified with the ring generated by $1_{\mathbb{K}}$. Then

$$\dim_{\Bbbk} I_t \;=\; \dim_{\mathbb{Q}}(I \cap \mathbb{Q}[x_0, \ldots, x_n])_t \;\geq\; \dim_{\mathbb{F}_p}(I_{\mathbb{F}_p})_t.$$

Here $I_{\mathbb{F}_p} \subseteq \mathbb{F}_p[x_0, \ldots, x_n]$ is the reduction modulo p of the ideal I. Checking that the upper bound (3) is attained is much easier over a finite field, and it guarantees the bound is attained over $\mathbb{Q}$, hence over $\Bbbk$. This leads to the following strategy for proving Theorem 5.1. First implement the expected Hilbert function and the expected gap size according to Conjecture 1 and Conjecture 2, here called `expectedHF`(n, r, t) and `expectedGapSize`(n, r). Next, execute the following steps for given (n, r):

1. Determine $d := \min\{\, t \mid h_S(t) \geq r \,\}$.
2. Sample r points $Z \subseteq \mathbb{P}^n(\mathbb{F}_p)$ (represented by homogeneous integer coordinates).
3. Calculate the ideal $I := I(Z)$ and set $J := \langle I_d \rangle_{\mathbb{F}_p[x_0,\ldots,x_n]}$.
4. Calculate the Hilbert function of S/J up to $d +$ `expectedGapSize`(n, r).
5. Check if the values match with `expectedHF`(n, r, t).

Proposition 5.2 and the preceding remark about reduction modulo p ensure that this procedure *proves* the validity of Conjecture 1 in the cases of Theorem 5.1. The following code in `Macaulay2`, assuming an implementation of `expectedGapSize` and `expectedHF`, demonstrates the procedure.

```
loadPackage "Points"
n = 2; r = 41;
d = 8; -- determined by (n,r)
I = points randomPointsMat(ZZ/1009[x_0..x_n], r);
e = expectedGapSize(n,r) -- 3
J = ideal select(first entries gens I, f -> degree f == {d});
hs = hilbertSeries(J, Order=>d+e+1) --
↪ 1 + 3T + 6T^2 + 10T^3 + 15T^4 + 21T^5 + 28T^6 + 36T^7 + 41T^8 + 43T^9 + 42T^10 + 41T^11
T = (ring hs)_0;
for t to d+e do
    assert (coefficient(T^t, hs) == expectedHF(n,r,t))
```

We briefly discuss some additional speed-ups. In the cases outside of the range (4), where Conjecture 1 is equivalent to the IGC, it is *much* faster to calculate `mingens`$(I(Z))$ and compare with the expected first graded Betti numbers $\beta_{1,d}$, $\beta_{1,d+1}$. Also, much computation time is spent computing the ideal of points. For large r, a significant speedup is obtained when using the methods implemented in `Points.m2` [SSS+].

We conclude with a variation on the computer experiment. Instead of computing ideals of (random) points one can also try to find monomial ideals certifying Conjecture 1. This method, for instance, can be used to prove the MRC in $\mathbb{P}^2$ [Sau85, GGR86]. An exhaustive search is possible in $\mathbb{P}^2$ for small values of r and leads to the following result:

Theorem 5.3. *Let $n = 2$, $S = \Bbbk[x, y, z]$.*

(i) *For $r = 18$ there is a unique, up to permutation of the variables, monomial ideal $I = \langle x^3y^2, y^3z^2, z^3x^2, x^2y^2z^2\rangle_S$ with Hilbert function (1), which satisfies Conjecture 1.*

(ii) *For $r \in \{25, 32, 33\}$ there are* no *monomial ideals satisfying Conjecture 1.*

6 Symmetric tensor decomposition

This final section discusses the role of chopped ideals in tensor decomposition algorithms. This was our original motivation and this project was initiated by a question encountered by one of the authors and Nick Vannieuwenhoven in [TV22]. The setting in that paper is slightly different because it studies algorithms for (ordinary) tensor decomposition. The same approach is classical in *symmetric tensor decomposition* or *Waring decomposition* [IK99, BCMT10]. A *Waring decomposition* of a homogeneous polynomial $F \in T = \Bbbk[y_0, \ldots, y_n]$ of degree D is an expression of F as a sum

$$F \;=\; c_1\,(z_1 \cdot y)^D \;+\; \cdots \;+\; c_r\,(z_r \cdot y)^D \tag{10}$$

of powers of linear forms. Here $c_i \in \Bbbk$ are constants and $z_i \cdot y = z_{i0}y_0 + \cdots + z_{in}y_n$. The *Waring rank* of F is the minimal number of summands r in such an expression.

Most decomposition algorithms aim to determine the vectors z_i up to scaling, and then solve a linear system to find the coefficients c_i. Therefore, it is natural to regard $Z = (z_1, \ldots, z_r)$ as a point in $(\mathbb{P}^n)^r$. A classical approach to compute Z uses *apolarity theory* and dates back to [Syl52]. We briefly recall the basics.

The ring $S = \Bbbk[x_0, \ldots, x_n]$ acts on the ring T by differentiation: if $g \in S$ and $F \in T$, then $g \bullet F = g(\partial_0, \ldots, \partial_n)F$ where $\partial_j = \frac{\partial}{\partial y_j}$. This action is graded. In particular, every $F \in T_D$ gives rise to a sequence of linear maps

$$\begin{aligned} C_F(d, D-d)\colon\; S_d &\longrightarrow T_{D-d} \\ g &\longmapsto g \bullet F, \end{aligned}$$

called the *Catalecticant maps* of F. Notice that $\operatorname{Ker} C_F(d, D-d) \subseteq S_d$ is a linear space of polynomials of degree d, and that $C_F(d, D-d) = 0$ if $d > D$. The kernels $\operatorname{Ker} C_F(d, D-d)$ are the homogeneous components of an ideal, called the *apolar ideal* of F, given by

$$\operatorname{Ann}(F) = \{\, f \in S \mid f \bullet F = 0 \,\}\,.$$

On the other hand, S can be naturally regarded as the homogeneous coordinate ring of $\mathbb{P}^n$. The classical *apolarity lemma* [BCC+18, Sec. 1, Lem. 5] states that F decomposes as in (10) if and only if the vanishing ideal $I(Z)$ of $Z = (z_1, \ldots, z_r)$ is contained in $\operatorname{Ann}(F)$.

It is usually hard to compute the ideal $I(Z)$ of a minimal Waring decomposition of F. However, in a restricted range, it turns out that its chopped ideal is generated by the graded component of $\mathrm{Ann}(F)$ in degree $d = \mathrm{reg}_{\mathrm{H}}(Z)$. In other words, the chopped ideal of Z can be computed via elementary linear algebra as kernel of the corresponding catalecticant map. This is known as the catalecticant method to determine a decomposition of F and it is the starting point of a number of more advanced Waring decomposition algorithms [BGI11, BCMT10, BT20, LMR23]. We record a consequence of [IK99, Thm. 2.6, Lem. 1.19].

Theorem 6.1. *Let $D \geq 2d$. If $F \in T_D$ is a general form of rank $r < h_S(d) - n$, then*

- *there is a unique Waring decomposition $Z \subseteq (\mathbb{P}^n)^r$ of length r and*
- $\mathrm{Ker}\, C_F(d, D-d) = I(Z)_d$ *generates the chopped ideal* $I(Z)_{\langle d \rangle}$.

This suggests a strategy for computing the Waring decomposition (10) of $F \in T_D$, compare [BCC⁺18, Alg. 2.82]:

1. Construct the catalecticant matrix $C_F(d, D-d)$, with $d = \lfloor \frac{D}{2} \rfloor$.
2. Compute a basis $f_1, \ldots, f_s$ for the kernel of $C_F(d, D-d)$ using linear algebra over $\Bbbk$,
3. Solve the polynomial system $f_1 = \cdots = f_s = 0$ on $\mathbb{P}^n$. Let $Z = (z_1, \ldots, z_r) \in (\Bbbk^{n+1})^r$ be the tuple of homogeneous coordinate vectors for the solutions.
4. Solve the linear equations (10) for $c_1, \ldots, c_r$.

When the rank of F is at most $h_S(\lfloor D/2 \rfloor) - (n+1)$, and under suitable genericity assumptions, Theorem 6.1 guarantees that this method computes the unique Waring decomposition of F. Moreover, if one knows $r < h_S(d) - n$ for some $d < D/2$, the approach can be made more efficient by computing a smaller catalecticant matrix.

Example 6.2. Consider a general ternary form of degree $D = 10$ with Waring rank $r = 18$,

$$F = (z_1 \cdot y)^{10} + \cdots + (z_{18} \cdot y)^{10}.$$

Here $y = (y_0, y_1, y_2)$ and each z_i has three coordinates as well. The catalecticant matrix $C_F(5, 5)$ is of size 21×21, and has rank 18. Its kernel consists of three ternary quintics in the variables x_0, x_1, x_2 passing through the 18 prescribed points $z_1, \ldots, z_{18}$. They generate the chopped ideal $I_{\langle 5 \rangle} = \langle I(Z)_5 \rangle$ investigated in Section 3.1. ◇

The main work in this strategy is step 3: solving the polynomial system $f_1 = \cdots = f_s = 0$. If $\Bbbk = \mathbb{C}$, two important strategies for doing this numerically are *homotopy continuation* [Tim21] and *numerical normal form methods* [Tel20]. We argue that in this setting it is natural to use the latter type of methods. Indeed, by construction, the system has $s = h_S(d) - r > n$ equations and $n+1$ variables, hence it is overdetermined. In homotopy continuation, this is typically dealt with by solving a square subsystem of n equations which has, by Bézout's theorem, $d^n > r$ solutions. These candidate solutions are filtered by checking if all remaining equations also vanish. However, computing all these d^n solutions becomes quickly infeasible. More refined algorithms using homotopy continuation are proposed in [BDHM17], but they rely on the knowledge of certain information on secant varieties which is out of reach with current methods. On the contrary, numerical normal form methods work directly with the overdetermined system, see [BT21, Sec. 4.4]. A second advantage is that, while homotopy continuation requires $\Bbbk = \mathbb{C}$, normal form methods work over any algebraically closed field $\Bbbk$.

Numerical normal form methods such as [BT21] and [Tel20, Sec. 4.5] compute the points z_i via the eigenvalues and/or eigenvectors of pairwise commuting *multiplication matrices*. These are in turn computed from a different matrix $M(d+e)$, called *Macaulay matrix*. Here $e \geq \gamma_{n,d}(r)$ is a positive integer for which $h_{S/I_{\langle d \rangle}}(d+e) = r$: the number of rows of $M(d+e)$ is $h_S(d+e)$, and its column span is $I(Z)_{d+e}$. In particular, upper bounds on the saturation gap allow one to work with the smallest admissible value e_0. We summarize the relation between Waring decomposition and Conjecture 2 as follows:

The complexity of computing multiplication matrices in our setting is governed by linear algebra computations with the Macaulay matrix $M(d+e_0)$, where $e_0 = \gamma_{n,d}(r)$.

To illustrate this punchline, we implemented the catalecticant algorithm in the Julia package `Catalecticant.jl`. It uses `EigenvalueSolver.jl`, a general purpose equation solver from [BT21]. Here is how to construct and decompose F from Example 6.2:

```
1 n = 2; D = 10; r = 18; # define the parameters
2 @polyvar y[1:n+1] # variables of F
3 Z = exp.(2*pi*im*rand(r, n+1)) # generate random points Z
4 F = sum((Z*y).^D)
5 c, linforms = waring_decomposition(F,y,r) # decompose F
```

Here line 3 draws the coordinates of the points Z uniformly from the unit circle in the complex plane; this avoids bad numerical behavior in the expansion $(z_i \cdot y)^D$. The output at line 5 is the pair of coefficients of the linear forms from (10). This method assumes Conjecture 2: `EigenvalueSolver.jl` constructs the Macaulay matrix $M(d+e_0)$, where $e_0 = \gamma_{n,d}(r)$ is the saturation gap predicted in Conjecture 2. In this specific case, we have $d + e_0 = 7$, as illustrated in Example 1.1

The code is available at https://mathrepo.mis.mpg.de/ChoppedIdeals/. It includes a file `examples.jl` which illustrates some other functionalities, such as computing Hilbert functions, catalecticant matrices and their kernel. Our code performs well, and may be of independent interest for Waring decomposition. On a 16 GB MacBook Pro with an Intel Core i7 processor working at 2.6 GHz, it computes the decomposition of a rank $r = 400$ form of degree $D = 12$ in $n + 1 = 6$ variables with 10 digits of accuracy within 25 seconds.

Future work

We conclude with some directions for future research. Chopped ideals are relevant for a large class of varieties, besides projective space. For instance, other types of tensor decomposition lead to points in multi-projective space [TV22]. One can also study ideals of points, and their chopped ideals, in arbitrary toric varieties, rational homogeneous varieties, or other varieties for which it makes sense to consider a multi-graded Hilbert function. This relates to decomposition algorithms and secant varieties as in [BB21, Sta23, Gał23].

Finally, it is possible to study chopped ideals for positive-dimensional varieties. For instance, there are 7 sextics passing through 11 general lines in $\mathbb{P}^3$. These generate a non-saturated chopped ideal, whose saturation is the vanishing ideal of the union of the lines, which has 4 additional generators in degree 7. For more general classes of varieties, there are several possible choices of chopped ideal to consider, and it would be interesting to explore generalizations of Conjecture 1.

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A On the missing sextic

Consider a partition of Z into two disjoint subsets Z_1, Z_2, each consisting of 9 points. Since $h_S(3) = 10$, there are two distinct cubics g_1, g_2 such that $I(Z_i)_3 = \langle g_i \rangle_{\Bbbk}$. We will prove that the sextic $g = g_1 g_2$ is not generated by the quintics f_0, f_1, f_2; in particular $g \notin (I_{\langle 5 \rangle})_6$ and $I(Z) = \langle f_0, f_1, f_2, g \rangle_S$. In order to prove this result, we introduce some geometric tools.

The first one is an elementary fact about fibers of a branched cover, which is at the foundation of monodromy techniques:

Proposition A.1. *Let X, Y be irreducible, reduced, quasi-projective varieties and let $f \colon X \to Y$ be a finite map. Let Z be a closed subvariety of X which intersects the generic fibers of f. Then $Z = X$.*

Proof. Without loss of generality assume f is surjective. Consider the graph Γ of f in $X \times Y$. That is, $\Gamma = \{ (x, y) \in X \times Y \mid y = f(x) \}$. Since f is a morphism and X is irreducible, Γ is closed in $X \times Y$ and irreducible. Moreover, since f is finite, $\dim X = \dim Y = \dim \Gamma$.

Let $\Gamma' = \Gamma \cap Z \times Y$, which is the graph of $f|_Z \colon Z \to Y$; since Z is closed in X, Γ' is closed in Γ. Since Z intersects the generic fiber of f, we deduce that $f|_Z$ is dominant. In particular, $\dim \Gamma' = \dim Z \geq \dim Y$ which in turn implies $\dim \Gamma' = \dim Y$. Therefore $\Gamma' \subseteq \Gamma$ is a closed subset, the two sets have the same dimension and Γ is irreducible; this implies $\Gamma' = \Gamma$. Applying the projection to the first factor of $X \times Y$, we conclude $Z = X$. $\square$

The second result is Proposition A.3, which consists in a generalization of [BL13, Lem. 8.1]. In order to prove it, we need the following version of Bertini's Theorem, derived from [Jou83, Thm. 6.3]:

Lemma A.2. *Let $X \subseteq \mathbb{P}^N$ be an irreducible projective variety with singular locus X_{sing}. Let J be a linear series on X with base locus $B \subseteq X$ and let $Y \in J$ be a general element. Then $Y \setminus (X_{\mathrm{sing}} \cup B)$ is smooth. Moreover, if $\dim J \geq 2$, then $Y \setminus B$ is irreducible.*

Proof. Write $m + 1 = \dim J$; the linear series on X defines a regular map $\varphi\colon X \setminus B \to \mathbb{P}^m$ and Y is the (closure of the) the generic fiber of this map. Consider an affine open cover of $\mathbb{P}^m$ with the property that every pair of points belongs to at least one affine open subset of the cover. The preimages of the open sets of this cover define a cover of $X \setminus B$ using open quasi-projective varieties. Any pair of points of $X \setminus B$ belongs to at least one quasi-projective variety of this cover.

On each of these open sets the statement is true by [Jou83, Thm. 6.3]. Since smoothness can be checked locally, this guarantees that $Y \setminus (X_{\mathrm{sing}} \cup B)$ is smooth. If $Y \setminus B$ were reducible, consider a quasi-projective open set of the cover which intersects two distinct irreducible components. Then [Jou83, Thm. 6.3, part 4] yields a contradiction. □

We now prove that a reduced 0-dimensional linear section of a linearly non-degenerate variety is itself non-degenerate. This is a higher codimension analog of [BL13, Lem. 8.1].

Proposition A.3. *Let $X \subseteq \mathbb{P}^N$ be an irreducible variety of dimension c not contained in a hyperplane. Let L be a linear space with $\operatorname{codim} L = c$. Suppose $X \cap L$ is a set of reduced points. Then $X \cap L$ is not contained in a hyperplane in L.*

Proof. Let $I(L) = \langle \ell_1, \ldots, \ell_c \rangle$ be the ideal of L. Observe that the the points of $X \cap L$ are smooth points of X. To see this, let $p \in X \cap L$ and consider the local ring $\mathcal{O}_{X,p}$; since $X \cap L$ is a set of reduced points, $I(L)_p \subseteq \mathcal{O}_{X,p}$ coincides with the maximal ideal in $\mathcal{O}_{X,p}$; in particular (the localizations of) $(\ell_1, \ldots, \ell_c)$ define a regular sequence in $\mathcal{O}_{X,p}$, showing that $\mathcal{O}_{X,p}$ is a regular local ring, and equivalently that p is smooth in X.

Let $L = L_c \subseteq L_{c-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathbb{P}^N$ be a general flag of linear spaces, with $\operatorname{codim} L_j = j$. For every j, define $X_j = X \cap L_j$; in particular $X_c = X \cap L$. For every $j = 0, \ldots, c-1$, we will show that X_j is irreducible and smooth away from the singular locus of X. We proceed by induction on j. The base case $j = 0$ is straightforward.

For $j \geq 1$, suppose X_{j-1} is irreducible and smooth away from the singular locus of X. Then $I(L)|_{L_{j-1}}$ defines a (non-complete) linear series on X_j. Since $L_j \subseteq L_{j-1}$ is general, Lemma A.2 guarantees that $X_j = X_{j-1} \cap L_j$ is smooth away from the singularities of X_{j-1} and the base locus of $I(L)$. Moreover, since $j \leq c-1$, $\dim X_{j-1} \geq 2$, hence X_j is irreducible except possibly for components supported in the base locus of $I(L)$. The base locus of $I(L)$ is $X \cap L$, which, as shown above, consists of smooth points of X. This guarantees that there are no embedded components nor singularities supported on the points of $X \cap L$. We conclude that for every $j = 0, \ldots, c-1$, X_j is irreducible and smooth away from X_{sing}.

An induction argument on $j = 0, \ldots, c-1$, with successive applications of [BL13, Lem. 8.1], shows that $X_{j+1} = X \cap L_{j+1}$ is linearly non-degenerate in L_{j+1}. In particular, $X \cap L_c = X_{c-1} \cap L$ is linearly non-degenerate in L. This concludes the proof. □

Finally, we will use that the degree of the variety

$$\mathcal{P}_{3,3} = \{\, g \in \mathbb{P}S_6 \mid g = g_1 g_2 \text{ for some } g_1, g_2 \in S_3 \,\} \subseteq \mathbb{P}S_6$$

is $\frac{1}{2}\binom{18}{9}$. This can be computed using elementary intersection theory; see, e.g., [EH16, Sec. 2.2.2] for a similar calculation.

Proposition A.4. *Let $Z \subseteq \mathbb{P}^2$ be a set of 18 general points. For every bipartition $Z = Z_1 \dot\cup Z_2$ of Z into two sets of 9 points, one has $g = g_1 g_2 \notin I(Z)_{\langle 5\rangle}$, where $I(Z_i) = \langle g_i \rangle_S$.*

Proof. The proof is structured as follows. We first show that there is some partition for which $g = g_1 g_2 \in I(Z) \setminus I(Z)_{\langle 5\rangle}$. Then, we use Proposition A.1 to show that the same must hold for all partitions.

Since Z is general, we have $\dim I(Z)_6 = 10$, $\dim(I(Z)_{\langle 5\rangle})_9$. Notice $\dim \mathcal{P}_{3,3} = 9 + 9 = 18 = \operatorname{codim}_{\mathbb{P}S_6} \mathbb{P}I(Z)_6$. Let $W = \mathcal{P}_{3,3} \cap \mathbb{P}I(Z)_6 \subseteq \mathbb{P}S_6$.

For every $g = g_1 g_2 \in W$, observe that $Z_i = Z \cap \{g_i = 0\}$ defines a bipartition of Z into two subsets of 9 points. Indeed, $Z = Z_1 \cup Z_2$, and no subset of 10 points in Z has a cubic equation because of the genericity assumption. On the other hand, any bipartition of $Z = Z_1 \dot\cup Z_2$ into two subsets of 9 points gives rise to an element $g = g_1 g_2 \in W$. By genericity, all these elements are distinct and they are smooth points of W. This shows that $W = \mathcal{P}_{3,3} \cap \mathbb{P}I(Z)_6$ is a set of $\frac{1}{2}\binom{18}{2} = \deg \mathcal{P}_{3,3}$ points. In particular, W is reduced.

By Proposition A.3, W is linearly non-degenerate in $\mathbb{P}I(Z)_6$, so it is not contained in the hyperplane $\mathbb{P}(I(Z)_{\langle 5\rangle})_6$. This shows that at least one element of W is not contained in the chopped ideal $I(Z)_{\langle 5\rangle}$.

We now show that $W \subseteq \mathbb{P}I(Z)_6 \setminus \mathbb{P}(I(Z)_{\langle 5\rangle})_6$. Consider the varieties:

$$\mathcal{Y} = \overline{\{\, (I(Z)_{\langle 5\rangle})_6 \in \mathrm{Gr}(9, S_6) \mid \ Z \subseteq \mathbb{P}^2 \text{ is a set of 18 points in general position } \}},$$

$$\mathcal{X} = \overline{\left\{ (Z_1, Z_2, g_1, g_2) \in (\mathbb{P}^2)^9 \times (\mathbb{P}^2)^9 \times \mathbb{P}S_3 \times \mathbb{P}S_3 \;\middle|\; \begin{array}{l} Z_1 \cap Z_2 = \emptyset \\ Z_1, Z_2, Z_1 \cup Z_2 \text{ in general position} \\ I(Z_i)_3 = \langle g_i \rangle \end{array} \right\}}$$

These are projective subvarieties of the Grassmannian $\mathrm{Gr}(9, S_6)$ and of the product $(\mathbb{P}^2)^9 \times (\mathbb{P}^2)^9 \times \mathbb{P}S_3 \times \mathbb{P}S_3$ respectively. Define the rational map

$$\varphi\colon \mathcal{X} \dashrightarrow \mathcal{Y}, \quad (Z_1, Z_2, g_1, g_2) \longmapsto (I(Z_1 \cup Z_2)_{\langle 5\rangle})_6.$$

Let $\mathcal{Z}$ be the subvariety of $\mathcal{X}$ defined by

$$\mathcal{Z} = \overline{\{\, (Z_1, Z_2, g_1, g_2) \mid g_1 g_2 \in (I(Z_1 \cup Z_2)_{\langle 5\rangle})_6 \,\}}.$$

If $W \cap \mathbb{P}((I(Z)_{\langle 5\rangle})_6) \neq \emptyset$, then $\mathcal{Z}$ would intersect the generic fiber of φ. In this case, Proposition A.1 guarantees $\mathcal{X} = \mathcal{Z}$. This implies that for every $(Z_1, Z_2, g_1, g_2) \in \mathcal{Z}$, we have $g_1 g_2 \in (I(Z_1 \cup Z_2)_{\langle 5\rangle})_6$. In other words $W \subseteq \mathbb{P}((I(Z)_{\langle 5\rangle})_6)$, which contradicts what we saw above. We conclude $W \cap \mathbb{P}((I(Z)_{\langle 5\rangle})_6) = \emptyset$. In other words, every $g = g_1 g_2$ satisfies $g \notin I(Z)_{\langle 5\rangle}$ as desired. $\square$

References

[AC22] E. Angelini and L. Chiantini. On the description of identifiable quartics. *Lin. and Mult. Algebra*, page 1–29, 2022. doi:10.1080/03081087.2022.2052004.

[Bal87] E. Ballico. Generators for the homogeneous ideal of s general points in $\mathbb{P}^3$. *J. Algebra*, 106(1):46–52, 1987. doi:10.1016/0021-8693(87)90020-2.

[BB21] W. Buczyńska and J. Buczyński. Apolarity, border rank, and multigraded Hilbert scheme. *Duke Math. J.*, 170(16):3659–3702, 2021. doi:10.1215/00127094-2021-0048.

[BCC+18] A. Bernardi, E. Carlini, M. V. Catalisano, A. Gimigliano, and A. Oneto. The hitchhiker guide to: Secant varieties and tensor decomposition. *Mathematics*, 6(12):314, 2018. doi:10.3390/math6120314.

[BCMT10] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. *Lin. Alg. Appl.*, 433(11-12):1851–1872, 2010. doi:10.1016/j.laa.2010.06.046.

[BDHM17] A. Bernardi, N. S. Daleo, J. D. Hauenstein, and B. Mourrain. Tensor decomposition and homotopy continuation. *Diff. Geom. Appl.*, 55:78–105, 2017. doi:10.1016/j.difgeo.2017.07.009.

[BGI11] A. Bernardi, A. Gimigliano, and M. Idà. Computing symmetric rank for symmetric tensors. *J. Symb. Comput.*, 46(1):34–53, 2011. doi:10.1016/j.jsc.2010.08.001.

[BL13] J. Buczyński and J. M. Landsberg. Ranks of tensors and a generalization of secant varieties. *Lin. Alg. Appl.*, 438(2):668–689, 2013. doi:10.1016/j.laa.2012.05.001.

[BT20] A. Bernardi and D. Taufer. Waring, tangential and cactus decompositions. *J. Math. Pures Appl.*, 143:1–30, 2020. doi:10.1016/j.matpur.2020.07.003.

[BT21] M. R. Bender and S. Telen. Yet another eigenvalue algorithm for solving polynomial systems. *arXiv:2105.08472*, 2021.

[Chi19] L. Chiantini. Hilbert Functions and Tensor Analysis. In *Quantum Physics and Geometry*, page 125–151. Springer, 2019. doi:10.1007/978-3-030-06122-7_6.

[EH16] D. Eisenbud and J. Harris. *3264 and All That - A Second Course in Algebraic Geometry*. Cambridge University Press, Cambridge, 2016.

[Eis95] D. Eisenbud. *Commutative Algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Eis05] D. Eisenbud. *The Geometry of Syzygies: A Second Course in Commutative Algebra and Algebraic Geometry*. Springer New York, 2005. doi:10.1007/0-387-26456-6_3.

[EM99] I. Z. Emiris and B. Mourrain. Matrices in elimination theory. *J. Symb. Comput.*, 28(1-2):3–44, 1999. doi:10.1006/jsco.1998.0266.

[EP96] D. Eisenbud and S. Popescu. Gale duality and free resolutions of ideals of points. *Inventiones mathematicae*, 136:419–449, 1996. doi:10.1007/s002220050315.

[EPSW02] D. Eisenbud, S. Popescu, F.-O. Schreyer, and C. Walter. Exterior algebra methods for the minimal resolution conjecture. *Duke Math. J.*, 112(2):379–395, 2002. doi:10.1215/S0012-9074-02-11226-5.

[Frö85] R. Fröberg. An inequality for Hilbert series of graded algebras. *Mathematica Scandinavica*, 56(2):117–144, 1985. URL: https://www.jstor.org/stable/24491560.

[Gał23] M. Gałązka. Multigraded apolarity. *Math. Nach.*, 296(1):286–313, 2023. doi:10.1002/mana.202000484.

[GGR86] A. V. Geramita, D. Gregory, and L. Roberts. Monomial ideals and points in projective space. *J. Pure Appl. Algebra*, 40:33–62, 1986. doi:10.1016/0022-4049(86)90029-0.

[GM84] A. V. Geramita and P. Maroscia. The ideal of forms vanishing at a finite set of points in $\mathbb{P}^n$. *J. Algebra*, 90(2):528–555, 1984. doi:10.1016/0021-8693(84)90188-1.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/. (version 1.17.1).

[Har92] J. Harris. *Algebraic geometry. A first course*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992.

[HS96] A. Hirschowitz and C. Simpson. La résolution minimale de l'idéal d'un arrangement général d'un grand nombre de points dans $\mathbb{P}^n$. *Inventiones mathematicae*, 126:467–503, 1996. doi:10.1007/s002220050107.

[IK99] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci*, volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[Jou83] J.-P. Jouanolou. *Théorèmes de Bertini et applications*, volume 42. Springer, 1983.

[Lan12] J. M. Landsberg. *Tensors: Geometry and Applications*, volume 128 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[LMR23] A. Laface, A. Massarenti, and R. Rischter. Decomposition algorithms for tensors and polynomials. *SIAM J. Appl. Alg. Geom.*, 7(1):264–290, 2023. doi:10.1137/21M1453712.

[Lor93] A. Lorenzini. The minimal resolution conjecture. *J. Algebra*, 156(1):5–35, 1993. doi:10.1006/jabr.1993.1060.

[Mat87] H. Matsumura. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1987. doi:10.1017/CBO9781139171762.

[Mig98] J. Migliore. *Introduction to liaison theory and deficiency modules*, volume 165. Springer Science & Business Media, 1998.

[MN02] J. Migliore and U. Nagel. Liaison and related topics: Notes from the Torino Workshop/School. *arXiv:0205161*, 2002.

[Rus16] F. Russo. *On the Geometry of Some Special Projective Varieties*. Springer International Publishing, 2016. doi:10.1007/978-3-319-26765-4.

[Sau85] T. Sauer. The number of equations defining points in general position. *Pacific J. Math.*, 120(1):199 – 213, 1985. doi:10.2140/pjm.1985.120.199.

[SSS$^+$] M. Stillman, G. G. Smith, S. A. Strømme, D. Eisenbud, F. Galetto, and J. W. Skelton. Points: sets of points. Version 3.0. A `Macaulay2` package available at https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages.

[Sta23] R. Staffolani. Schur apolarity. *J. Symb. Comp.*, 114:37–73, 2023. doi:10.1016/j.jsc.2022.04.017.

[Syl52] J. J. Sylvester. On the principles of the calculus of forms. *Cambridge and Dublin Math. J.*, 7:52–97, 1852.

[Tel20] S. Telen. *Solving Systems of Polynomial Equations*. PhD thesis, KU Leuven, Leuven, Belgium, 2020. Available at https://simontelen.webnode.page/publications/.

[Tim21] S. Timme. *Numerical nonlinear algebra*. PhD thesis, Technische Universitaet Berlin (Germany), 2021. Available at https://sascha.timme.xyz/.

[TV22] S. Telen and N. Vannieuwenhoven. A normal form algorithm for tensor rank decomposition. *ACM Trans. on Math. Soft.*, 48(4):1–35, 2022. doi:10.1145/3555369.

[Wal95] C. H. Walter. The minimal free resolution of the homogeneous ideal of s general points in $\mathbb{P}^4$. *Math. Zeit.*, 219:231–234, 1995. doi:10.1007/BF02572362.

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