60 minutes of local cohomology

Seminar on Nonlinear Algebra

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How many equations define a curve in $\mathbb{P}^3$ set-theoretically?

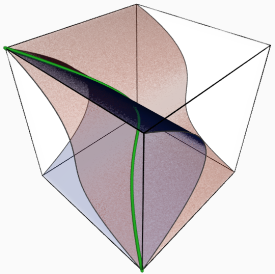

Figure 1: The twisted cubic C_3.

$$I(C_3) = \langle \text{minors of } \left(\begin{smallmatrix} x & y & z \\ y & z & w \end{smallmatrix}\right) \rangle$$
$$= \langle xz - y^2,\ yw - z^2,\ xw - yz \rangle$$
$$C_3 = \mathrm{V}\begin{pmatrix} xz - y^2, \\ z(yw - z^2) - w(xw - yz) \end{pmatrix}$$

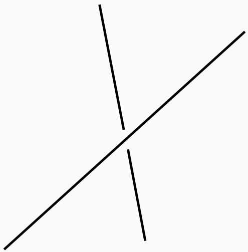

Figure 2: Two skew lines $L \cup L'$.

$$I(L \cup L') = \langle x, y \rangle \cap \langle z, w \rangle$$
$$= \langle xz,\ xw,\ yz,\ yw \rangle$$
$$L \cup L' = \mathrm{V}(xz,\ yw,\ xw + yz)$$

Possible with 2 equations?

Local cohomology – an apéritif

Let R be a Noetherian ring and $\mathfrak{a} \subseteq R$ an ideal. There exist additive functors of R-modules $\mathrm{H}^i_{\mathfrak{a}}(-)$, $i = 0, 1, \ldots$ with the following properties:

1. $(\mathrm{H}^i_I(-))_i$ is a universal δ-functor: $0 \to K \to L \to M \to 0$ induces LES

$$0 \to \mathrm{H}^0_{\mathfrak{a}}(K) \to \mathrm{H}^0_{\mathfrak{a}}(L) \to \mathrm{H}^0_{\mathfrak{a}}(M) \to \mathrm{H}^1_{\mathfrak{a}}(K) \to \mathrm{H}^1_{\mathfrak{a}}(L) \to \mathrm{H}^1_{\mathfrak{a}}(M) \to \mathrm{H}^2_{\mathfrak{a}}(K) \to \cdots$$

2. Depends only on radical: $\mathrm{H}^i_{\mathfrak{a}}(-) = \mathrm{H}^i_{\sqrt{\mathfrak{a}}}(-)$
3. Lower bounds on no. of generators: If $\mathfrak{a} = \langle f_1, \ldots, f_s \rangle$, then $\mathrm{H}^i_{\mathfrak{a}}(-) = 0$ for $i > s$
4. Detects algebraic invariants of M: dimension, depth, regularity, ...

First definition: Derived functors

- Consider the left-exact $\mathfrak{a}$-*power torsion* functor

$$\Gamma_{\mathfrak{a}}(M) := \left\{ m \in M \;\middle|\; \mathfrak{a}^k m = 0,\ k \gg 0 \right\} \subseteq M$$

- Example: $\Gamma_{\mathfrak{m}}(\Bbbk[x_0, \ldots, x_n]/I) = I^{\mathrm{sat}}/I$
- Observe: $\sqrt{\mathfrak{a}}^r \subseteq \mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$ for some r, hence $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\sqrt{\mathfrak{a}}}(M)$
- $\mathrm{H}^i_{\mathfrak{a}}$ are the right derived functors of $\Gamma_{\mathfrak{a}}$: For M choose an injective resolution $0 \to M \to E^0 \to E^1 \to \ldots$, then

$$\mathrm{H}^i_{\mathfrak{a}}(M) := \mathrm{H}^i(\Gamma_{\mathfrak{a}}(E^\bullet)) = \frac{\operatorname{Im}\left(\Gamma_{\mathfrak{a}}(E^{i-1}) \to \Gamma_{\mathfrak{a}}(E^i)\right)}{\operatorname{Ker}\left(\Gamma_{\mathfrak{a}}(E^i) \to \Gamma_{\mathfrak{a}}(E^{i+1})\right)}$$

- Alternative abstract nonsense definition

$$\Gamma_{\mathfrak{a}}(M) \cong \operatorname{colim}_k \operatorname{Hom}_R(R/\mathfrak{a}^k, M) \qquad \rightsquigarrow \qquad \mathrm{H}^i_{\mathfrak{a}}(M) \cong \operatorname{colim}_k \operatorname{Ext}^i_R(R/\mathfrak{a}^k, M)$$

Example: Modules over a principal ideal domain

Let $R = \Bbbk[x]$, $\mathfrak{a} = \langle x \rangle$, M finite R-module

- By the structure theorem, suffices to consider $M = R$ and $M = R/\langle f \rangle$
- R has an injective resolution $0 \to R \to K \to K/R \to 0$, $K = \mathrm{Frac}(R)$

$$\mathrm{H}^0_{\langle x \rangle}(R) = 0, \qquad \mathrm{H}^1_{\langle x \rangle}(R) = \Gamma_{\langle x \rangle}(K/R) = R[x^{-1}]/R, \qquad \mathrm{H}^i_{\langle x \rangle}(R) = 0, \quad i \geq 2$$

- If $f = x^n g$, then $\mathrm{H}^0_{\langle x \rangle}(R/\langle f \rangle) = \langle g \rangle / \langle f \rangle \cong R/\langle x^n \rangle$
- The long exact sequence to $0 \to R \xrightarrow{\cdot f} R \to R/\langle f \rangle \to 0$ yields

$$0 \longrightarrow \mathrm{H}^0_{\langle x \rangle}(R/\langle f \rangle) \longrightarrow R[x^{-1}]/R \xrightarrow{\cdot f} R[x^{-1}]/R \longrightarrow \mathrm{H}^1_{\langle x \rangle}(R/\langle f \rangle) \longrightarrow 0$$

- Multiplication with f is surjective, hence $\mathrm{H}^i_{\langle x \rangle}(R/\langle f \rangle) = 0$ for $i \geq 1$

Second definition: Čech complex

Pick a generating set $\mathfrak{a} = \langle x_1, \ldots, x_n \rangle$. For $J \subseteq \{1, \ldots, n\}$ let $M[x_J^{-1}]$ be the localization of M at $x_J := \prod_{j \in J} x_i$. Define $(C^\bullet, d)$ via

$$C^i(x_1, \ldots, x_n; M) := \bigoplus_{\#J = i} M[x_J^{-1}], \qquad d \colon M[x_J^{-1}] \ni m_J \mapsto \sum_{j \notin J} (-1)^{\#\{J < j\}} m_{J \cup \{j\}}$$

This complex has terms $C^0, \ldots, C^n$

$$0 \longrightarrow M \xrightarrow{d^0} \bigoplus_{j=1}^{n} M[x_j^{-1}] \xrightarrow{d^1} \bigoplus_{\#J=2} M[x_J^{-1}] \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} M[x_{\{1,\ldots,n\}}^{-1}] \longrightarrow 0$$

Theorem

$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i(C^\bullet(x_1, \ldots, x_n; M))$ *for* $i \geq 0$.

Consequence: $\mathrm{H}^i_{\langle x_1, \ldots, x_n \rangle}(M) = 0$ for $i > n$

Example: The polynomial ring

- For $n = 1$ one has $\mathrm{H}^1_{\langle x \rangle}(\Bbbk[x]) \cong x^{-1}\Bbbk[x^{-1}]$, since the Čech complex is

$$0 \longrightarrow \Bbbk[x] \longrightarrow \Bbbk[x, x^{-1}] \longrightarrow 0$$

- For $n = 2$ we have

$$0 \longrightarrow \Bbbk[x, y] \xrightarrow{\binom{1}{1}} \Bbbk[x^{\pm 1}, y] \oplus \Bbbk[x, y^{\pm 1}] \xrightarrow{(1,-1)} \Bbbk[x^{\pm 1}, y^{\pm 1}], \longrightarrow 0$$

  here $\mathrm{H}^1_{\langle x,y \rangle}(\Bbbk[x, y]) = 0$ and $\mathrm{H}^2_{\langle x,y \rangle}(\Bbbk[x, y]) \cong \langle x^{-1}, y^{-1} \rangle \Bbbk[x^{-1}, y^{-1}]$

Theorem

Let $S = \Bbbk[x_1, \ldots, x_n] = M$, $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. Then all local cohomology modules vanishes except

$$\mathrm{H}^n_{\mathfrak{m}}(S) \cong \langle x_1^{-1}, \ldots, x_n^{-1} \rangle \Bbbk[x_1^{-1}, \ldots, x_n^{-1}] \cong S[-n]^{\vee}.$$

A Mayer-Vietoris-like sequence

Theorem

Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals, then there is a long exact sequence

$$\cdots \to \mathrm{H}^{i-1}_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \mathrm{H}^{i}_{\mathfrak{a}+\mathfrak{b}}(M) \to \mathrm{H}^{i}_{\mathfrak{a}}(M) \oplus \mathrm{H}^{i}_{\mathfrak{b}}(M) \to \mathrm{H}^{i}_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \mathrm{H}^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M) \to \dots$$

- *Proof idea:* Apply $\mathrm{Ext}^i_R(-, M)$ to the short exact sequence

$$0 \longrightarrow R/(\mathfrak{a}^k \cap \mathfrak{b}^k) \longrightarrow R/\mathfrak{a}^k \oplus R/\mathfrak{b}^k \longrightarrow R/(\mathfrak{a}^k + \mathfrak{b}^k) \longrightarrow 0,$$

  then take the appropriate colimit
- Example: $R = \Bbbk[x, y, z, w]$, $\mathfrak{a} = \langle x, y\rangle$, $\mathfrak{b} = \langle z, w\rangle$, $I = \mathfrak{a} \cap \mathfrak{b}$, then

$$\cdots \to \underbrace{\mathrm{H}^3_{\langle x,y\rangle}(M) \oplus \mathrm{H}^3_{\langle z,w\rangle}(M)}_{=0} \to \mathrm{H}^3_I(M) \xrightarrow{\cong} \mathrm{H}^4_{\mathfrak{m}}(M) \to \underbrace{\mathrm{H}^4_{\langle x,y\rangle}(M) \oplus \mathrm{H}^4_{\langle z,w\rangle}(M)}_{=0} \to \cdots$$

Putting the pieces together

Assume $I = \langle xz,\ xw,\ yz,\ yw\rangle = \sqrt{\langle f_1, f_2\rangle}$ for $f_1, f_2 \in R = \Bbbk[x, y, z, w]$

- Being two-generated, one has $\mathrm{H}^3_{\langle f_1, f_2\rangle}(-) = 0$
- Local cohomology only depends on the radical, so $\mathrm{H}^3_I(-) = 0$
- The previous Mayer-Vietoris example showed $\mathrm{H}^3_I(-) \cong \mathrm{H}^4_{\mathfrak{m}}(-)$
- The calculation of local cohomology of polynomial rings showed $\mathrm{H}^4_{\mathfrak{m}}(R) \neq 0$. ↯

Open problem: Is the following rational quartic curve

$$C = \{\ (s^4 : s^3t : st^3 : t^4)\ |\ (s : t) \in \mathbb{P}^1(\mathbb{C})\ \}$$

a set-theoretic complete intersection? Here $\mathrm{H}^3_{I(C)}(-) = 0$. 🤔

Let's go deeper!

From now on let M be a finitely generated R-module with $\mathfrak{a}M \subsetneq M$

- $\operatorname{depth}_{\mathfrak{a}}(M)$ is the maximal length of a M-regular sequence $f_1, \ldots, f_d \in I$.
- $\dim(M)$ is the dimension $\dim R/\operatorname{Ann}_R(M)$
- $\operatorname{depth}_{\mathfrak{a}}(M) \leq \dim(M)$
- If R is local/graded, then the *Cohen-Macaulay-property* is $\operatorname{depth}_{\mathfrak{m}}(M) \overset{!}{=} \dim(M)$

Theorem

1. $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for* $i < \operatorname{depth}_{\mathfrak{a}}(M)$, $\mathrm{H}^{\operatorname{depth}(M)}_{\mathfrak{a}}(M) \neq 0$.
2. $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for* $i > \dim(M)$, $\mathrm{H}^{\dim(M)}_{\mathfrak{m}}(M) \neq 0$ *if* R *is local or graded.*

Consequence: M is Cohen-Macaulay iff there is a single nonzero $\mathrm{H}^d_{\mathfrak{m}}(M)$

Local vs. sheaf cohomology

Let R be a standard-graded Noetherian ring, $\mathfrak{m} := R_{>0}$ and M finite graded.

- $\mathrm{H}^i_{\mathfrak{m}}(M)$ is a graded module, $\mathrm{H}^i_{\mathfrak{m}}(M)_n$ is finitely generated over R_0
- Let $X = \mathrm{Proj}(R)$, $\widetilde{M}$ the sheaf associated to M, $\widetilde{M}(d) = \widetilde{M[d]} = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$
- The direct sum of the twists form a graded R-module $\bigoplus_{d\in\mathbb{Z}} \mathrm{H}^i(X, \widetilde{M}(d))$

Theorem (Comparison theorem)

There is an exact sequence of graded R-modules

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \hookrightarrow M \to \bigoplus_d \Gamma(X, \widetilde{M}(d)) \to \mathrm{H}^1_{\mathfrak{m}}(M) \to 0$$

and isomorphisms $\bigoplus_d \mathrm{H}^i(X, \widetilde{M}(d)) \cong \mathrm{H}^{i+1}_{\mathfrak{m}}(M)$, $i \geq 1$.

Serre vanishing: $\mathrm{H}^i_{\mathfrak{m}}(M)_n = 0$ *for* $i \geq 0$ *and* $n \gg 0$.

The Hilbert polynomial – demystified

Now assume $R_0 = \Bbbk$, i.e. R is an affine graded $\Bbbk$-algebra

- Let $h_M(d) = \dim_\Bbbk M_d$ be the Hilbert function and P_M the Hilbert polynomial
- The Euler characteristic of a sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \mathrm{h}^i(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim_\Bbbk \mathrm{H}^i(X, \mathcal{F}).$$

- $P_M(d) = \chi(\widetilde{M}(d))$ for all $d \in \mathbb{Z}$ and $\chi(\widetilde{M}(d)) = \mathrm{h}^0(\widetilde{M}(d))$ for $d \gg 0$
- Using the comparison theorem, we obtain the formula

$$\begin{aligned} h_M(d) &= \dim_\Bbbk \mathrm{H}^0_\mathfrak{m}(M)_d + \mathrm{h}^0(\widetilde{M}(d)) - \dim_\Bbbk \mathrm{H}^1_\mathfrak{m}(M)_d \\ &= \chi(\widetilde{M}(d)) - \sum_{i \geq 1} (-1)^i \mathrm{h}^i(\widetilde{M}(d)) + \sum_{i=0,1} (-1)^i \dim_\Bbbk \mathrm{H}^i_\mathfrak{m}(M)_d \\ &= P_M(d) + \sum_{i \geq 0} (-1)^i \dim_\Bbbk \mathrm{H}^i_\mathfrak{m}(M)_d \end{aligned}$$

Castelnuovo-Mumford regularity

- M admits a *minimal graded free resolution* ($\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$)

$$0 \longrightarrow F_\delta \xrightarrow{\varphi_\delta} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0, \qquad F_i = \bigoplus_j R[-j]^{\beta_{i,j}}$$

- The exponents $\beta_{i,j} = \dim_{\Bbbk} \operatorname{Tor}_i^R(M, \Bbbk)_j$ are the *graded Betti numbers* of M
- $\operatorname{reg}_{\mathrm{CM}}(M) := \max \{ i + j \mid \beta_{i,j} \neq 0 \}$ is the *Castelnuovo-Mumford regularity* of M
- For a graded module N let $\operatorname{end}(N) := \sup \{ d \mid N_d \neq 0 \}$

Theorem

$\operatorname{reg}_{\mathrm{CM}}(M) = \max_i \operatorname{end}(\mathrm{H}^i_{\mathfrak{m}}(M)) + i.$

If M is Cohen-Macaulay of dim. d, then $\operatorname{reg}_{\mathrm{CM}}(M) = \operatorname{end}(\mathrm{H}^d_{\mathfrak{m}}(M)) + d$

Hilbert regularity over $S = \Bbbk[x_0, \ldots, x_n]$

- Let $\operatorname{reg}_{\mathrm{H}}(M) := \min \{ d \mid h_M(j) = P_M(j) \text{ for all } j \geq d \}$
- A free resolution gives a way to calculate $h_M(d)$:

$$0 \longrightarrow F_\delta \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \quad \rightsquigarrow \quad h_M(d) = \sum_{i=0}^{\delta} \sum_j (-1)^i \beta_{i,j} \cdot h_S(d-j)$$

- $h_S(d-j)$ agrees with the polynomial $\binom{d-j+n}{n}$ as soon as $d \geq j - n$
- Estimating this yields a bound on the Hilbert regularity

$$\operatorname{reg}_{\mathrm{H}}(M) \leq \max_{\beta_{i,j} \neq 0} j - n \leq \operatorname{reg}_{\mathrm{CM}}(M) + \delta - n = \operatorname{reg}_{\mathrm{CM}}(M) - \operatorname{depth}_{\mathfrak{m}}(M) + 1$$

using the Auslander-Buchsbaum formula $\operatorname{depth}_{\mathfrak{m}}(M) + \operatorname{pdim}_S(M) = \operatorname{depth}_{\mathfrak{m}}(S)$
- If M is CM, then equality holds, as $h_M(j) - P_M(j) = (-1)^d \dim_{\Bbbk} \mathrm{H}^d_{\mathfrak{m}}(M)_j$

Example: Ideals of points

Let I be an ideal with $Z = V(I) \subseteq \mathbb{P}^n$ finite of length r

- $\dim S/I = 1$, so $\mathrm{H}^i_{\mathfrak{m}}(S/I) = 0$ for $i \geq 2$; $\mathrm{H}^0_{\mathfrak{m}}(S/I) = I^{\mathrm{sat}}/I$
- The long exact sequence associated to $0 \to I^{\mathrm{sat}}/I \to S/I \to S/I^{\mathrm{sat}} \to 0$ yields

$$\cdots \longrightarrow \underbrace{\mathrm{H}^1_{\mathfrak{m}}(I^{\mathrm{sat}}/I)}_{=0} \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(S/I) \xrightarrow{\cong} \mathrm{H}^1_{\mathfrak{m}}(S/I^{\mathrm{sat}}) \longrightarrow \underbrace{\mathrm{H}^2_{\mathfrak{m}}(I^{\mathrm{sat}}/I)}_{=0} \longrightarrow \cdots$$

- The comparison sequence for $J = I^{\mathrm{sat}}$ reads

$$0 \longrightarrow S/J \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_Z(d)) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(S/J) \longrightarrow 0$$

- Middle term has constant dim. r $\quad\rightsquigarrow\quad$ $\operatorname{end} \mathrm{H}^1_{\mathfrak{m}}(S/J) + 1 = \operatorname{reg}_{\mathrm{H}}(S/J)$

Theorem (Regularity for one-dimensional ideals)

$\operatorname{reg}_{\mathrm{CM}}(S/I) = \max\{\operatorname{reg}_{\mathrm{H}}(S/I) - 1, \operatorname{reg}_{\mathrm{H}}(S/I^{\mathrm{sat}})\}$

Consequence: Minimal generators of J live in degree $\leq \operatorname{reg}_{\mathrm{H}}(S/J) + 1$

- Local duality: $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{Ext}_S^{n+1-i}(M, S[-n-1])^\vee$
  This generalizes to more general graded or local rings
- McMullen's Upper bound theorem: *Cyclic polytopes have the largest number of faces among convex polytopes with the same dimension and number of vertices*
  Stanley's proof of the simplicial version involves Cohen-Macaulayness of the Stanley-Reisner ring, local cohomology being an important ingredient
- D-modules: The local cohomology modules are D-modules in a natural way. As such, they are *finitely generated* (!) and better suited for computation.
  Also, GKZ A-hypergeometric systems!

Thank you! Questions?