Complexity of Polynomial Subalgebras and their Initial Algebras

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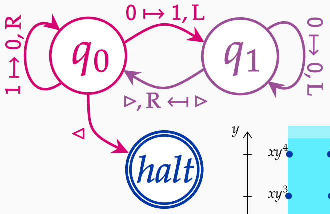

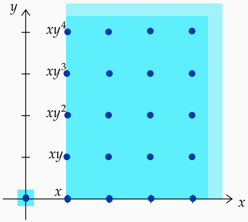

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Computational mathematicians always have problems

Definition (Computational problem, Decision problem)

A **computational problem** consists of an input, e.g. a tuple of data, and a question or expected output. A **decision problem** has output `yes` or `no`.

- Input/output encoded over **finite alphabet $\boldsymbol{\Sigma}$**, $\Sigma^* := \{\text{strings over } \Sigma\}$
- Decision problems are just subsets $A \subseteq \Sigma^*$ (the "`yes`"-instances)

Definition (Ideal membership problem $\texttt{IdealMem}_K$)

Input: $f_1, \ldots, f_s, g \in \boldsymbol{R} := K[x_1, \ldots, x_n]$

Question: $g \in \langle f_1, \ldots, f_s \rangle_R$? (Decision problem)

Output: $h_1, \ldots, h_s \in R$ with $g = h_1 f_1 + \cdots + h_s f_s$ (Representation problem)

Definition (Turing machine)

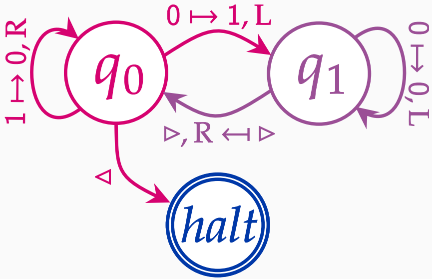
A **deterministic Turing machine** M (DTM) consists of

i) a finite set of **states** $\boldsymbol{Q}$, including an initial state q_0 and final states $F \subseteq Q$;

ii) a **tape alphabet** $\boldsymbol{\Gamma}$ containing the in/output alphabets and a blank $\square \in \Gamma$;

iii) a **transition function** $\boldsymbol{\delta} \colon (Q \setminus F) \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$.

$$\begin{pmatrix} \text{current state,} \\ \text{read tape symbol} \end{pmatrix} \mapsto \begin{pmatrix} \text{next state,} \\ \text{overwrite symbol,} \\ \text{move left/right} \end{pmatrix}$$



- ▷ DTMs “roughly” equivalent to computers
- ▷ steps ≈ time, tape ≈ memory

Through time and space

Definition (TIME and SPACE)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function $\geq \log n$.

i) $\mathrm{TIME}(f) = \{$decision prob. $A \mid \exists$DTM M deciding $w \in A$ in $O(f(|w|))$ steps$\}$

ii) $\mathrm{SPACE}(f) = \{A \mid \exists$DTM M deciding $w \in A$ using $O(f(|w|))$ cells$\}$

$$\begin{aligned} & \mathrm{P} = \textstyle\bigcup_k \mathrm{TIME}(n^k) && \overset{?}{\subseteq} && \mathrm{NP} = \textstyle\bigcup_k \mathrm{NTIME}(n^k) \\ \subseteq \quad & \mathrm{PSPACE} = \textstyle\bigcup_k \mathrm{SPACE}(n^k) && \subsetneq && \mathrm{EXPSPACE} = \textstyle\bigcup_k \mathrm{SPACE}(2^{n^k}) \end{aligned}$$

Theorem (Hermann 1926, Mayr & Meyer 1982, Mayr 1989)

i) *If* $g = h_1 f_1 + \cdots + h_s f_s$, *then* $\exists (h_i)_i$ *with* $\deg h_i \leq \deg g + (s \cdot \max_i \deg f_i)^{2^n}$.

ii) $\mathtt{IdealMem}_{\mathbb{Q}} \in \mathrm{EXPSPACE}$. *One can compute some* $(h_i)_i$ *in space* $2^{O(|w|)}$.

For sake of completeness

Definition (Karp-reduction, hardness & completeness)

Let $A \subseteq \Sigma^*$, $B \subseteq \Delta^*$ be decision problems.

i) $A \leq_{\mathrm{m}}^{\mathrm{P}} B$ if there is a "simple" function $f \colon \Sigma^* \to \Delta^*$ with $w \in A \Leftrightarrow f(w) \in B$.

ii) B is **hard** for a complexity class $\mathcal{C}$ if $A \leq_{\mathrm{m}}^{\mathrm{P}} B$ for all $A \in \mathcal{C}$.

iii) B is **complete** for a complexity class $\mathcal{C}$ if $B \in \mathcal{C}$ and hard for $\mathcal{C}$.

- ▷ Reduction embeds problem A into problem B, "A is at most as difficult as B"
- ▷ Cook-Levin theorem: `3SAT` is NP-complete; stepping stone for hardness results

Theorem (Mayr & Meyer 1982, Mayr 1989)

i) *Hermann's degree bound $O((sd)^{2^n})$ for certificates $(h_i)_i$ is sharp.*

ii) $\texttt{IdealMem}_{\mathbb{Q}}$ *is $\mathrm{EXPSPACE}$-complete, even for binomial ideals.*

The scary doubly-exponential examples

Theorem (Dubé 1990, Kühnle & Mayr 1996)

Let $I = \langle f_1, \ldots, f_s \rangle_{K[x_1,\ldots,x_n]}$ *be an ideal and* $d = \max_i \deg f_i$*. The reduced Gröbner basis* $G = \{g_i\}_i$ *of* I *(w.r.t. an arbitrary monomial order) has degree*

$$\deg g_i \leq 2\left(\frac{d^2}{2} + d\right)^{2^{n-1}}.$$

One can enumerate the reduced Gröbner basis in exponential working space.

Theorem (Huynh 1986, my MA thesis 2022)

i) *There are ideals in* $K[x_1, \ldots, x_n]$ *generated by* $O(n)$ *polynomials of degree* $O(1)$*, whose reduced Gröbner basis has at least* 2^{2^n} *elements and degree* $\geq 2^{2^n}$*.*

ii) *Membership in the reduced Gröbner basis is* $\mathrm{EXPSPACE}$*-complete.*

Your focus determines your reality

Theorem (Mayr 1989, 1997)

$\texttt{IdealMem}_{\mathbb{Q}}$ *restricted to homogeneous polynomials is* PSPACE*-complete.*

- Gröbner bases can still be doubly-exponential even for homogeneous ideals
- Deciding whether $1 \in \langle f_1, \ldots, f_s \rangle_R$ (the "Nullstellensatz") is also in PSPACE, in fact low in the Polynomial Hierarchy (though at least NP-hard)
- Bounding the number of variables also drops the complexity to PSPACE
- There are also dimension-dependent degree bounds available
- The complexity of computing Gröbner bases seems to be linked to its Castelnuovo-Mumford regularity [Bayer & Mumford 1993]

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Subgalgebra Analogue to Membership Problem for Ideals (SAMPI)

Definition (Subalgebra membership problem $\mathtt{AlgMem}_K$)

Input: $f_1, \dots, f_s, g \in R = K[x_1, \dots, x_n]$

Question: $g \in K[f_1, \dots, f_s]$? (Decision problem)

Output: $p \in K[t_1, \dots, t_s]$ with $g = p(f_1, \dots, f_s)$ (Certification problem)

Some questions:

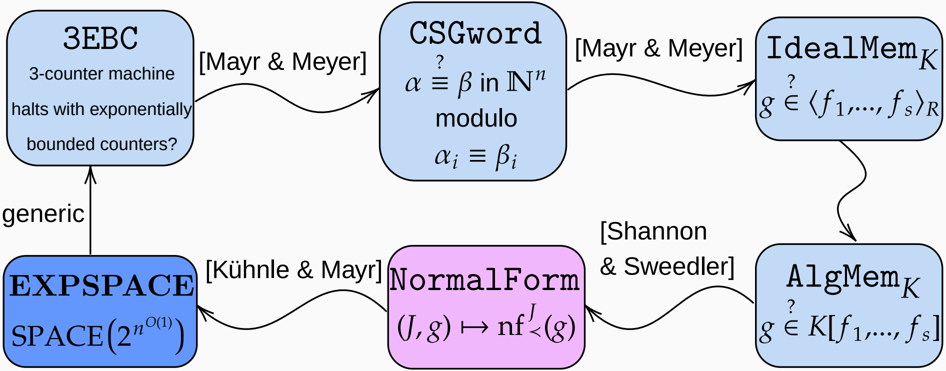
i) Degree bounds on p depending on $n, s, \deg f_i$?

ii) Upper and lower bounds on complexity of $\mathtt{AlgMem}_{\mathbb{Q}}$? Related to $\mathtt{IdealMem}_{\mathbb{Q}}$?

iii) Easier when the polynomials are homogeneous? Or monomials? Or n bounded?

iv) The analogue to Gröbner bases for ideals are SAGBI bases for subalgebras. What is the complexity of SAGBI bases?

A chain of reductions



Subalgebra membership using normal forms

- Given $f_1, \ldots, f_s, g \in K[x_1, \ldots, x_n]$, want to check if $g \in K[f_1, \ldots, f_n]$
- Consider the ideal $J = \langle f_1 - t_1, \ldots, f_s - t_s \rangle \subseteq K[\boldsymbol{x}, t_1, \ldots, t_s]$
- Let $\prec$ be a mon. order on $K[\boldsymbol{x}, \boldsymbol{t}]$ such that $x_i \succ \boldsymbol{t}^\alpha$ for all $x_i, \boldsymbol{t}^\alpha$, e.g. $\prec_{\text{lex}}$
- The normal form $\mathrm{nf}_\prec^J(g)$ is the unique $g' \in g + J$ such that no term in g' is divisible by the leading term of any element of J

Theorem (Shannon & Sweedler 1986, attributed to Spear)

$g \in K[f_1, \ldots, f_s]$ *if and only if* $p := \mathrm{nf}_\prec^J(g) \in K[\boldsymbol{x}, \boldsymbol{t}]$ *is in* $K[\boldsymbol{t}]$.
In this case, considering p *as a polynomial in* $t_1, \ldots, t_s$*, one has* $g = p(f_1, \ldots, f_s)$.

$\rightsquigarrow$ Reduces subalgebra membership to normal form calculation

The upper bound

Theorem (K. 2025)

$\texttt{AlgMem}_{\mathbb{Q}}$ is in $\mathrm{EXPSPACE}$. A certificate $p \in \mathbb{Q}[t_1, \ldots, t_s]$ can be computed using $2^{O(|w|)}$ working space.

Proof. Combine the previous elimination method with the exponential working space algorithm for normal forms by [Kühnle & Mayr 1996]. □

- Careful analysis reveals that the homogeneous problem is in PSPACE
- We also get a degree bound for the certificate using the Dubé bound:

Theorem (K. 2025)

If $g \in K[f_1, \ldots, f_s]$, $e := \deg g$, then there is a p with $p(f_1, \ldots, f_s) = g$ of degree

$$\deg p \leq e + \left(\left(\tfrac{1}{2} d^{2s^2} + d \right)^{2^n} + 1 \right)^{(n+s)^2+1} e^{n+s} \approx d^{O((n+s)^4 2^n)} e^{n+s}.$$

The exponential space lower bound

Lemma (K. 2025)

Let $f_1, \ldots, f_s, g \in R = K[x_1, \ldots, x_n]$*, then the following are equivalent:*

i) $g \in \langle f_1, \ldots, f_s \rangle_R$;
ii) $ug \in A := K[x_1, \ldots, x_n, uf_1, \ldots, uf_s] \subseteq R[u]$.

The minimal degree of $p \in K[t_1, \ldots, t_{n+s}]$ *with* $p(x_1, \ldots, uf_s) = ug$ *is one less than the minimal degree of a representation* $\max_i \deg h_i$*.The minimal number of terms of* p *coincides with the minimal total number of terms of* $h_1, \ldots, h_s$.

Theorem (K. 2025)

i) $\texttt{IdealMem}_{\mathbb{Q}} \leq_m^{\mathrm{P}} \texttt{AlgMem}_{\mathbb{Q}}$, *thus* $\texttt{AlgMem}_{\mathbb{Q}}$ *is* $\mathrm{EXPSPACE}$*-complete.*
ii) *Similar for homogen. polynomials,* $\texttt{AlgMem}_{\mathbb{Q}}(\text{homog})$ *is* PSPACE*-complete.*

What if I don't care about computational complexity?

Corollary (Worst-case examples for ideal membership)

For every n, there exists polynomials $f_1, \dots, f_s, g \in K[x_1, \dots, x_{O(n)}]$, $s \in O(n)$, such that

- $\deg f_i, \deg g \leq 6$,
- *each f_i, g has at most two terms (single variable or binomial),*
- *$g \in K[f_1, \dots, f_s]$, but every $p \in K[t_1, \dots, t_s]$ with $p(f_1, \dots, f_s) = g$ has degree and number of terms at least 2^{2^n}.*

If the f_i, g are homogeneous (degree $O(n)$), then one can still archieve 2^n terms.

Proof. Build counter machine as a commutative semigroup, embed as previously!

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The McNugget problem

Theorem (K. 2025)

$\texttt{IdealMem}_{\mathbb{Q}}$ *restricted to monomial algebras is* NP*-complete.*
This is still true for square-free monomials or the univariate case[1]*.*

- Here p can be chosen to be a monomial, this reduces to a problem in $(\mathbb{N}^n, +)$
- The univariate case is "exactly" the NP-complete change-making problem

$$x^{43} \overset{?}{\in} \mathbb{Q}[x^6, x^9, x^{20}] \quad \Leftrightarrow \quad 43 = 6a + 9b + 20c,\ a, b, c \in \mathbb{N}$$

- Problem is in NP, one can easily verify p; hardness from combinatorics

[1] But only with binary exponent encoding: $|\texttt{enc}(x^e)| \approx \log_2 e$

SAGBI bases are complicated ...

Definition (Initial algebra, SAGBI basis)

Given monomial order $\prec$ and subalgebra $A \subseteq K[x_1, \ldots, x_n]$, the **initial algebra** is

$$\mathrm{in}_\prec(A) := K[\{\, \mathrm{in}_\prec(g) \mid g \in A \setminus 0 \,\}]$$

A **SAGBI basis** of A is a set $S \subseteq A$ whose initial monomials generate $\mathrm{in}_\prec(A)$.

- Not every subalgebra $K[f_1, \ldots, f_s] \subseteq K[\boldsymbol{x}]$ has a finitely gen'd initial algebra

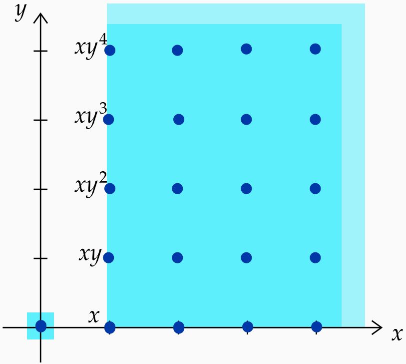
$$A = K[x, xy - y^2, xy^2], \qquad \rightsquigarrow \qquad \mathrm{in}_\prec(A) = K[x, xy, xy^2, xy^3, xy^4, \ldots]$$

- No known general criterion on finiteness of SAGBI bases
- Conjecture: The finiteness problem is compuationally hard (Undecidable?)

Theorem (Robbiano & Sweedler 1990)

$\texttt{SAGBIfinite}_K$ *is semi-decidable using the subduction algorithm.*

... but may have interesting structure?



$\text{in}_<(A) = 1K + xK[x,y]$

Definition (Affine-linear set, semilinear set)

i) An **affine-linear set** $X \subseteq \mathbb{Z}^n$ has the form $X = v_0 + \langle v_1, \ldots, v_m \rangle_{\mathbb{N}}$, $v_i \in \mathbb{Z}^n$.

ii) A **semilinear set** $X \subseteq \mathbb{Z}^n$ is a finite union of affine-linear sets.

The semilinearity conjecture

- Conjecture: The initial monomials of a finitely generated subalgebra $A \subseteq K[x_1, \ldots, x_n]$ form a semilinear set (if $\prec$ is reasonable, say $\prec_{\mathrm{lex}}$)
- Always true if $\mathrm{in}_\prec(A)$ is finitely generated (even linear set)
- All known examples seem to have this structure
- Not true for "wild" monomimal orders

Theorem (K. & Reinke 2025+)

If $A = K[x_1, \ldots, x_n]^G$, $G \leq \mathfrak{S}_n$ and $\prec$ is a rational weight order, then the semilinearity conjecture holds for A.

- Idea: Semilinear sets are exactly sets in $\mathbb{N}^n$ described by Presburger formulas
- Can define initial algebra membership here as Presburger formula

Hope: Semilinear presentation can aid algebra, geometry & computation!

Thank you! Questions?