Computational Complexity of Polynomial Subalgebras

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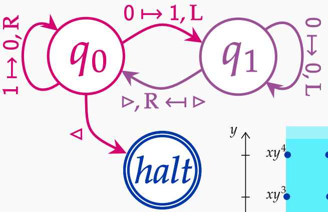
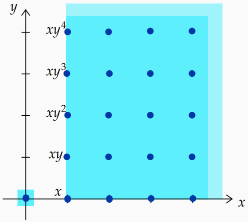




The dessert menu

Computational complexity

Subalgebra membership

Monomial and initial algebras

Mathematicians always have problems

Definition (Computational problem, Decision problem)

A **computational problem** consists of an input, e.g. a tuple of data, and a question or expected output. A **decision problem** has output `yes` or `no`.

- Input/output encoded over **finite alphabet $\boldsymbol{\Sigma}$**, $\Sigma^* := \{\text{strings over } \Sigma\}$
- Decision problems are just subsets $A \subseteq \Sigma^*$ (the "`yes`"-instances)

Definition (Ideal membership problem $\texttt{IdealMem}_K$)

Input: $f_1, \ldots, f_s, g \in \boldsymbol{R} := K[x_1, \ldots, x_n]$

Question: $g \in \langle f_1, \ldots, f_s \rangle_R$? (Decision problem)

Output: $h_1, \ldots, h_s \in R$ with $g = h_1 f_1 + \cdots + h_s f_s$ (Representation problem)

The Turing model of computation

Definition (Turing machine)

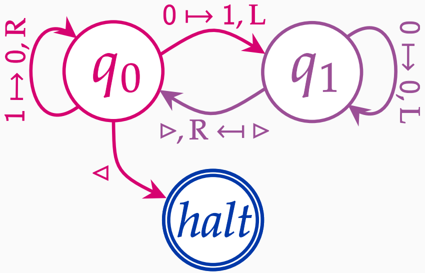
A **deterministic Turing machine** M (DTM) consists of

i) a finite set of **states** $\boldsymbol{Q}$, including an initial state q_0 and final states $F \subseteq Q$;

ii) a **tape alphabet** $\boldsymbol{\Gamma}$ containing the in/output alphabet;

iii) a **transition function** $\boldsymbol{\delta}\colon Q \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$.

$$\begin{pmatrix} \text{current state,} \\ \text{read tape symbol} \end{pmatrix} \overset{\delta}{\mapsto} \begin{pmatrix} \text{next state,} \\ \text{overwrite symbol,} \\ \text{move left/right} \end{pmatrix}$$



▷ steps ≈ time, tape ≈ memory

Through time and space

Definition (TIME and SPACE)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function $\geq \log n$.

i) $\mathrm{TIME}(f) = \{$decision prob. $A \mid \exists$DTM M deciding $w \in A$ in $O(f(|w|))$ steps$\}$

ii) $\mathrm{SPACE}(f) = \{A \mid \exists$DTM M deciding $w \in A$ using $O(f(|w|))$ cells$\}$

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k) \overset{?}{\subseteq} \mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k)$$
$$\subseteq \mathrm{PSPACE} = \bigcup_k \mathrm{SPACE}(n^k) \subsetneq \mathrm{EXPSPACE} = \bigcup_k \mathrm{SPACE}(2^{n^k})$$

Theorem (Hermann 1926, Mayr & Meyer 1982, Mayr 1989)

i) *If* $g = h_1 f_1 + \cdots + h_s f_s$, *then* $\exists (h_i)_i$ *with* $\deg h_i \leq \deg g + (s \cdot \max_i \deg f_i)^{2^n}$.

ii) $\texttt{IdealMem}_{\mathbb{Q}} \in \mathrm{EXPSPACE}$. *One can compute some* $(h_i)_i$ *in space* $2^{O(|w|)}$.

For sake of completeness

Definition (Karp-reduction, hardness & completeness)

Let $A \subseteq \Sigma^*$, $B \subseteq \Delta^*$ be decision problems.

i) $A \leq_{\mathbf{m}}^{\mathbf{P}} B$ if there is a "simple" function $f\colon \Sigma^* \to \Delta^*$ with $w \in A \Leftrightarrow f(w) \in B$.

ii) B is **hard** for a complexity class $\mathcal{C}$ if $A \leq_{\mathrm{m}}^{\mathrm{P}} B$ for all $A \in \mathcal{C}$.

iii) B is **complete** for a complexity class $\mathcal{C}$ if $B \in \mathcal{C}$ and hard for $\mathcal{C}$.

- Reduction embeds problem A into problem B, "A is at most as difficult as B"
- Cook-Levin theorem: `3SAT` is NP-complete; stepping stone for hardness results

Theorem (Mayr & Meyer 1982, Mayr 1989)

i) *Hermann's degree bound* $O((sd)^{2^n})$ *for certificates* $(h_i)_i$ *is sharp.*

ii) $\mathtt{IdealMem}_{\mathbb{Q}}$ *is* $\mathrm{EXPSPACE}$*-complete, even for binomial ideals.*

The scary doubly-exponential examples

Theorem (Dubé 1990, Kühnle & Mayr ISSAC'96)

Let $I = \langle f_1, \ldots, f_s \rangle_{K[x_1,\ldots,x_n]}$ be an ideal and $d = \max_i \deg f_i$. The reduced Gröbner basis $G = \{g_i\}_i$ of I (w.r.t. any monomial order) has degree

$$\deg g_i \leq 2\left(\frac{d^2}{2} + d\right)^{2^{n-1}}.$$

One can enumerate the reduced Gröbner basis in exponential working space.

Theorem (Huynh 1986, my MA thesis 2022)

i) *There are ideals in $K[x_1, \ldots, x_n]$ generated by $O(n)$ polynomials of degree $O(1)$, whose reduced Gröbner basis has at least 2^{2^n} elements and degree $\geq 2^{2^n}$.*

ii) *Membership in the reduced Gröbner basis is $\mathrm{EXPSPACE}$-complete.*

The (not so) ideal world

Theorem (Mayr 1989, 1997)

$\texttt{IdealMem}_{\mathbb{Q}}$ *restricted to homogeneous polynomials is* PSPACE*-complete.*

- Gröbner bases can still be doubly-exponential even for homogeneous ideals
- Deciding whether $1 \in \langle f_1, \ldots, f_s \rangle_R$ (the "Nullstellensatz") is also in PSPACE, in fact low in the Polynomial Hierarchy (though at least NP-hard)
- Bounding the number of variables also drops the complexity to PSPACE
- There are dimension-dependent degree bounds available [Mayr & Ritscher 2013]
- The complexity of computing Gröbner bases seems to be linked to its Castelnuovo-Mumford regularity [Bayer & Mumford 1993]

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Subalgebra membership

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Subgalgebra Analogue to Membership Problem for Ideals (SAMPI)

Definition (Subalgebra membership problem $\texttt{AlgMem}_K$)

Input: $f_1, \ldots, f_s, g \in K[x_1, \ldots, x_n]$

Question: $g \in K[f_1, \ldots, f_s]$? (Decision problem)

Output: $p \in K[t_1, \ldots, t_s]$ with $g = p(f_1, \ldots, f_s)$ (Certification problem)

Some questions:

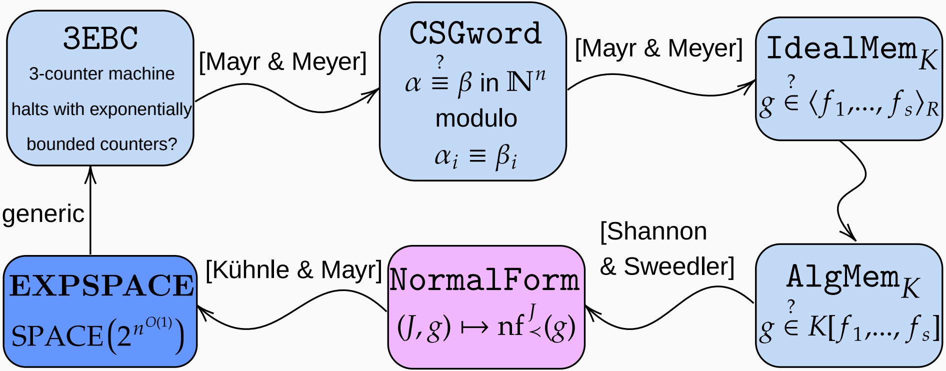
i) Degree bounds on p depending on $n, s, \deg f_i$?

ii) Upper and lower bounds on complexity of $\texttt{AlgMem}_{\mathbb{Q}}$? Related to $\texttt{IdealMem}_{\mathbb{Q}}$?

iii) Easier when the polynomials are homogeneous? Or monomials? Or n bounded?

iv) The analogue to Gröbner bases for ideals are SAGBI bases for subalgebras. What is the complexity of SAGBI bases?

A chain of reductions



Subalgebra membership using normal forms

- Given $f_1, \ldots, f_s, g \in K[x_1, \ldots, x_n]$, want to check if $g \in K[f_1, \ldots, f_n]$
- Consider the ideal $J = \langle f_1 - t_1, \ldots, f_s - t_s \rangle \subseteq K[\boldsymbol{x}, t_1, \ldots, t_s]$
- Let $\prec$ be a mon. order on $K[\boldsymbol{x}, \boldsymbol{t}]$ such that $x_i \succ \boldsymbol{t}^\alpha$ for all $x_i, \boldsymbol{t}^\alpha$, e.g. $\prec_{\text{lex}}$
- The normal form $\mathrm{nf}_\prec^J(g)$ is the unique $g' \in g + J$ such that no term in g' is divisible by the leading term of any element of J

Theorem (Shannon & Sweedler 1986, attributed to Spear)

$g \in K[f_1, \ldots, f_s]$ *if and only if* $p := \mathrm{nf}_\prec^J(g) \in K[\boldsymbol{x}, \boldsymbol{t}]$ *is in* $K[\boldsymbol{t}]$.
In this case, considering p *as a polynomial in* $t_1, \ldots, t_s$*, one has* $g = p(f_1, \ldots, f_s)$.

$\rightsquigarrow$ Reduces subalgebra membership to normal form calculation

The upper bound

Theorem (K. 2025)

$\texttt{AlgMem}_{\mathbb{Q}}$ *is in* $\mathrm{EXPSPACE}$ *and* $\texttt{AlgMem}_{\mathbb{Q}}(\text{homog})$ *is in* PSPACE.
A certificate $p \in \mathbb{Q}[t_1, \ldots, t_s]$ *can be computed using* $2^{O(|w|)}$ *working space.*

Proof idea. Combine the previous elimination method with the exponential working space algorithm for normal forms by [Kühnle & Mayr 1996]. □

- Careful analysis reveals that the bounded variable case is also in PSPACE
- We also get a degree bound for the certificate using the Dubé bound:

Theorem (K. 2025)

If $g \in K[f_1, \ldots, f_s]$, $e := \deg g$, *then there is a* p *with* $p(f_1, \ldots, f_s) = g$ *of degree*

$$\deg p \le e + \left(\left(\tfrac{1}{2} d^{2s^2} + d \right)^{2^n} + 1 \right)^{(n+s)^2+1} e^{n+s} \approx d^{O((n+s)^4 2^n)} e^{n+s}.$$

The exponential space lower bound

Lemma (K. 2025)

Let $f_1, \dots, f_s, g \in R = K[x_1, \dots, x_n]$*, then the following are equivalent:*

i) $g \in \langle f_1, \dots, f_s \rangle_R$;
ii) $ug \in A \coloneqq K[x_1, \dots, x_n, uf_1, \dots, uf_s] \subseteq R[u]$.

The minimal degree of $p \in K[t_1, \dots, t_{n+s}]$ *with* $p(x_1, \dots, uf_s) = ug$ *is one less than the minimal degree of a representation* $\max_i \deg h_i$*.The minimal number of terms of* p *coincides with the minimal total number of terms of* $h_1, \dots, h_s$*.*

Theorem (K. 2025)

i) $\texttt{IdealMem}_{\mathbb{Q}} \leq_m^{\mathrm{P}} \texttt{AlgMem}_{\mathbb{Q}}$*, thus* $\texttt{AlgMem}_{\mathbb{Q}}$ *is* $\mathrm{EXPSPACE}$*-complete.*
ii) *Similar for homogen. polynomials,* $\texttt{AlgMem}_{\mathbb{Q}}(\text{homog})$ *is* PSPACE*-complete.*

Worst-case examples for subalgebra membership

Corollary (The Mayr–Meyer algebras)

For every n, there exists polynomials $f_1, \ldots, f_s, g \in K[x_1, \ldots, x_{O(n)}]$, $s \in O(n)$, such that

- $\deg f_i, \deg g \leq 6$,
- *each f_i, g has at most two terms (single variable or binomial),*
- *$g \in K[f_1, \ldots, f_s]$, but every $p \in K[t_1, \ldots, t_s]$ with $p(f_1, \ldots, f_s) = g$ has degree and number of terms at least 2^{2^n}.*

If the f_i, g are homogeneous (degree $O(n)$), then one can still archieve 2^n terms.

Idea. Build counter machine as a commutative semigroup, embed into subalgebra

The binary counting subalgebra

- Tape content $b_1 \cdots b_n \in \{0,1\}^n$, state q_i, head position j, $\hat{=}\ q_i h_j x_{1,b_1} \cdots x_{n,b_n}$

$$
\begin{aligned}
\mathcal{T} = \{q_0, q_1\} \,\dot\cup\, \{h_0, \ldots, h_n\}, \quad \boldsymbol{x} = \{x_{1,0}, x_{1,1}, \ldots, x_{n,0}, x_{n,1}\}, \\
\mathcal{R} &:= \{\, q_0 h_i x_{i,0} - q_1 h_{i-1} x_{i,1} \mid 1 \le i \le n \,\} \\
&\quad \cup \{\, q_0 h_i x_{i,1} - q_0 h_{i+1} x_{i,0} \mid 1 \le i \le n-1 \,\} \\
&\quad \cup \{\, q_1 h_i x_{i,0} - q_1 h_{i-1} x_{i,0} \mid 1 \le i \le n \,\} \\
&\quad \cup \{\, q_1 h_0 - q_0 h_1 \,\}, \\
A &:= \mathbb{K}[f_1, \ldots, f_{5n}] := \mathbb{K}[\mathcal{R} \cup \boldsymbol{x}], \\
g &:= q_0 h_1 x_{1,0} \cdots x_{n,0} - q_0 h_n x_{1,0} \cdots x_{n-1,0} x_{n,1}.
\end{aligned}
$$

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The McNugget problem

Theorem (K. 2025)

$\texttt{IdealMem}_{\mathbb{Q}}$ restricted to monomial algebras is NP-complete.
This is still true if $d \leq n$ or the univariate case[1].

- Here p can be chosen to be a monomial, this reduces to a problem in $(\mathbb{N}^n, +)$
- The univariate case is "exactly" the NP-complete change-making problem

$$x^{43} \overset{?}{\in} \mathbb{Q}[x^6, x^9, x^{20}] \quad \Leftrightarrow \quad 43 = 6a + 9b + 20c,\ a, b, c \in \mathbb{N}$$

- Problem is in NP, one can easily verify p; hardness from combinatorics/ILPs

[1] But only with binary exponent encoding: $|\texttt{enc}(x^e)| \approx \log_2 e$. With unary encoding in TC^0.

SAGBI bases are complicated . . .

Definition (Initial algebra, SAGBI basis)

Given monomial order $\prec$ and subalgebra $A \subseteq K[x_1, \ldots, x_n]$, the **initial algebra** is

$$\mathrm{in}_\prec(A) := K[\{\, \mathrm{in}_\prec(g) \mid g \in A \setminus 0 \,\}]$$

A **SAGBI basis** of A is a set $S \subseteq A$ whose initial monomials generate $\mathrm{in}_\prec(A)$.

- Not every subalgebra $K[f_1, \ldots, f_s] \subseteq K[\boldsymbol{x}]$ has a finitely gen'd initial algebra

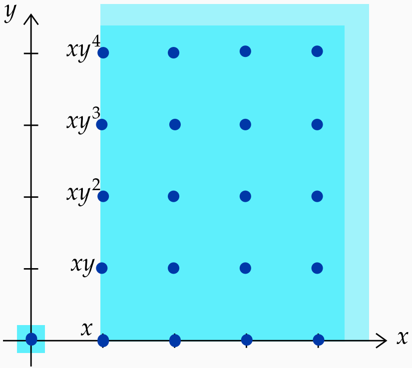
$$A = K[x, xy - y^2, xy^2], \qquad \rightsquigarrow \qquad \mathrm{in}_\prec(A) = K[x, xy, xy^2, xy^3, xy^4, \ldots]$$

- No known general criterion on finiteness of SAGBI bases
- Conjecture: The finiteness problem is computationally hard

Theorem (Robbiano & Sweedler 1990)

$\mathtt{SAGBIfinite}_K$ *is semi-decidable using the subduction algorithm.*

. . . but may have interesting structure?



$$\mathrm{in}_<(A) = 1K + xK[x,y]$$
$$= \{0\} \cup \left(e_1 + \mathbb{N}^2\right)$$

Definition (Affine-linear set, semilinear set)

i) An **affine-linear set** $X \subseteq \mathbb{Z}^n$ has the form $X = v_0 + \langle v_1, \ldots, v_m \rangle_{\mathbb{N}}$, $v_i \in \mathbb{Z}^n$.

ii) A **semilinear set** $X \subseteq \mathbb{Z}^n$ is a finite union of affine-linear sets.

The semilinearity conjecture

Semilinearity conjecture (K. & Reinke 2025+)

The initial monomials of a finitely generated subalgebra form a semilinear set.

- Clearly true if $\mathrm{in}_\prec(A)$ is finitely generated (even linear set)
- All known examples seem to have this structure
- Not true for “wild” monomimal orders, need “reasonable” orders

Theorem (K. & Reinke 2025+)

Let $G \leq \mathrm{GL}(\mathbb{Z}^n)$ be a finite group, $M \subseteq \mathbb{Z}^n$ an affine semigroup invariant under G and $\prec$ a rational weight order. Then the semilin. conjecture holds for $A = K[M]^G$.

- Idea: Semilinear sets are exactly sets in $\mathbb{N}^n$ described by Presburger formulas
- Can define initial algebra membership here as Presburger formula

Hope: Decide `SAGBIfinite` using effective semilinear presentation of $\mathrm{in}_\prec(A)$!

Thank you! Questions?