Computation and Complexity in Algebraic Geometry

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

Doctoral Defense

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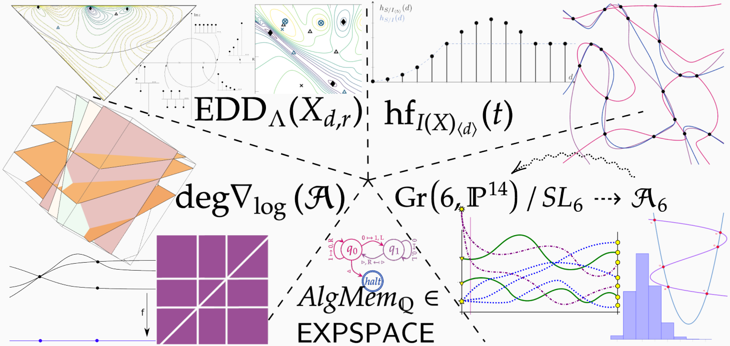
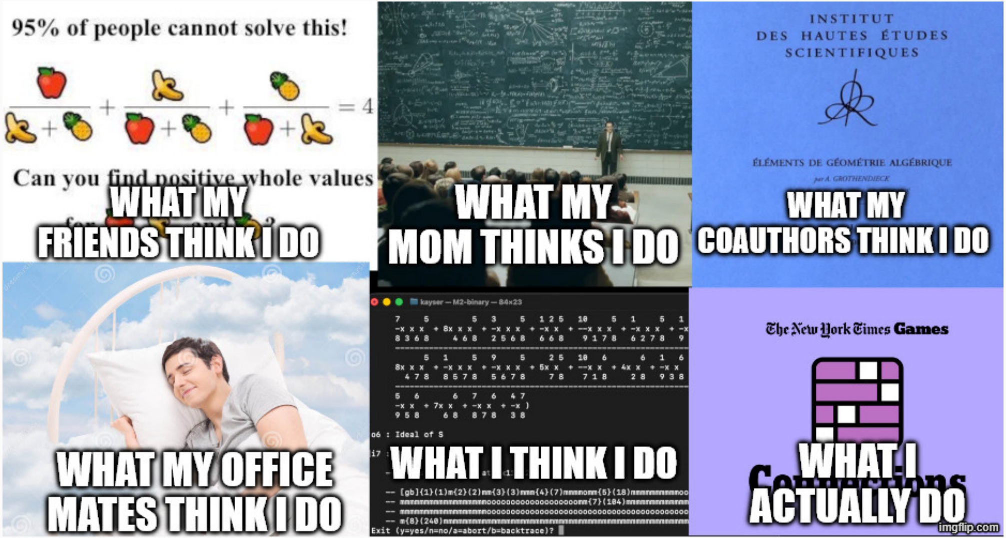
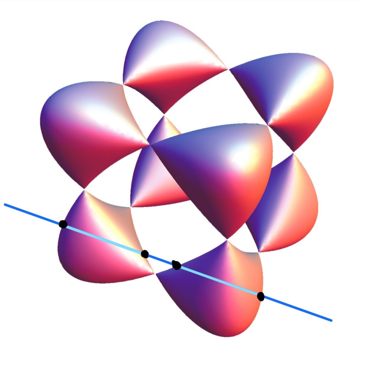
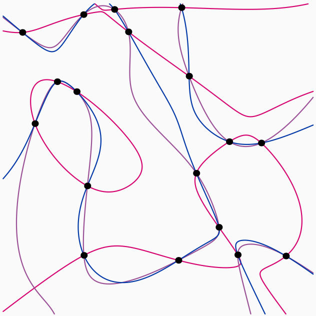


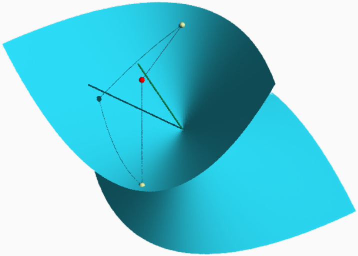
What does a math PhD student even do?



What is Algebraic Geometry?



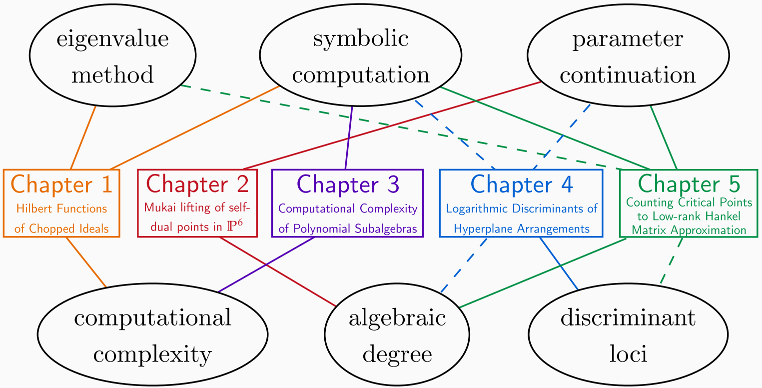




- ▷ Study of points, curves, surfaces. . . defined by polynomials, e.g. $x^2 + y^2 - 1 = 0$
- ▷ Ask about dimension, degree, equations, singularities, distances, families, . . .
- ▷ Interplay between algebra of defining equations and geometry of its variety

What is Computation and Complexity?

- Computation: Numerically determine solutions to polynomial equations, symbolic manipulation of ideals and other algebraic structures, sampling, . . .
- Complexity: Number of solutions, discriminant loci, algorithm runtime, . . .



$$\mathrm{EDD}_\Lambda(X_{d,r})$$

$$\mathrm{hf}_{I(X)_{\langle d\rangle}}(t)$$

$$\deg \nabla_{\log}(\mathcal{A})$$

$$\mathrm{Gr}\left(6, \mathbb{P}^{14}\right) / SL_6 \dashrightarrow \mathcal{A}_6$$

$$\mathit{AlgMem}_{\mathbb{Q}} \in \mathsf{EXPSPACE}$$

Rabbits everywhere!

- Consider the evolution of a model population of rabbits $\hat{y} = (1, 1, 2, 3, 5, 8, 13)^\mathsf{T}$
- Does this data sequence obey a simple law? Yes, $\hat{y}_i + \hat{y}_{i+1} - \hat{y}_{i+2} = 0$
- How do we find this linear recurrence relation $a = (1, 1, -1)^\mathsf{T}$? Linear algebra:

$$H_2(\hat{y}) \cdot a = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \\ 5 & 8 & 13 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \stackrel{!}{=} 0$$

- Is there a closed formula? Yes, Moivre–Binet's formula:
- Let $\varphi = \frac{1+\sqrt{5}}{2}$, $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$ be the roots of the quadratic equation $a(z) = 1 + z - z^2$, then

$$\hat{y}_i = \frac{\varphi^i - \overline{\varphi}^i}{\sqrt{5}}$$

Realization of linear time-invariant difference equations

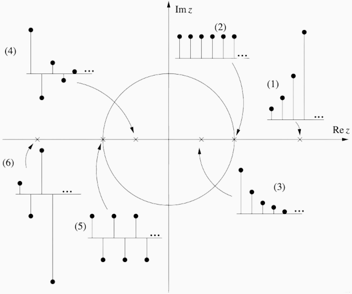
$$a_0\hat{y}_i + a_1\hat{y}_{i+1} + \cdots + a_r\hat{y}_{i+r} = 0, \qquad i = 0, \ldots, d-r$$

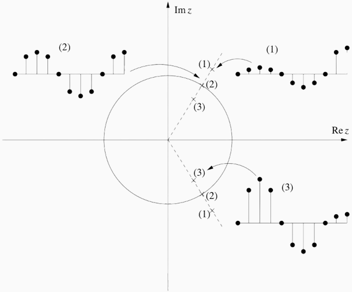
- Discrete-time (physical) system generating signals $y = (y_0, y_1, \ldots, y_d)^\mathsf{T} \in \mathbb{R}^{d+1}$
- Explain observed data with a mathematical model
- Impose model class: *autonomous LTI models of finite order*
- $\hat{y}$ "model compliant" data if satisfies difference equation a of given order r

$$\underbrace{\begin{bmatrix} a_0 & a_1 & \cdots & a_r & & & \\ & a_0 & a_1 & \cdots & a_r & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & a_0 & a_1 & \cdots & a_r \end{bmatrix}}_{=:\, T_d(a) \text{ Toeplitz matrix } (d-r+1)\times(d+1)} \begin{pmatrix} \hat{y}_0 \\ \vdots \\ \hat{y}_d \end{pmatrix} = \underbrace{\begin{bmatrix} \hat{y}_0 & \hat{y}_1 & \cdots & \hat{y}_r \\ \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{d-r} & \hat{y}_{d-r+1} & \cdots & y_d \end{bmatrix}}_{=:\, H_r(\hat{y}) \text{ Hankel matrix } (d-r+1)\times(r+1)} \begin{pmatrix} a_0 \\ \vdots \\ a_r \end{pmatrix} \overset{!}{=} 0$$

Roots of $a(z) = \sum_{i=0}^{r} a_i z^i$ determine dynamics of model

- Each root λ generates exponential sequence $\mathrm{vand}(\lambda) = (1, \lambda, \lambda^2, \ldots, \lambda^d)^\mathsf{T}$
- Magnitude of λ's determines growth or decay, argument determines phase





- Model-compliant data is linear combination of these modes

$$\hat{y} = \sum_{\lambda} c_\lambda \cdot \mathrm{vand}(\lambda) = \Big[\sum_{\lambda} c_\lambda \cdot \lambda^k \Big]_{k=0}^{d}$$

Exact realization vs. least squares realization

- $\hat{y}$ satisfies LTI difference equation iff $\operatorname{rk} H_r(\hat{y}) \leq r$, all such $\hat{y}$ form a variety

$$\mathcal{X}_{d,r} := \left\{ \hat{y} \in \mathbb{C}^{d+1} \;\middle|\; \operatorname{rk} H_r(\hat{y}) = \operatorname{rk} \begin{bmatrix} \hat{y}_0 & \hat{y}_1 & \cdots & \hat{y}_r \\ \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_{r+1} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{y}_{d-r} & \hat{y}_{d-r+1} & \cdots & y_d \end{bmatrix} \leq r \right\}$$

- Identify model a (difference equation) as kernel of Hankel matrix
- Real world scenario: Don't have access to $\hat{y}$, measure noisy $y = \hat{y} + \varepsilon$

$\rightsquigarrow$ y never satisfies a difference equation exactly, $\operatorname{rk} H_r(y) = r + 1$ almost surely

- Fix a 2-norm on $\mathbb{R}^{d+1}$, $Q_\Lambda(y) = \frac{1}{2}\|y\|^2 = \frac{1}{2} y^\mathsf{T} \Lambda y$
- Constrained least squares realization problem

$$\underset{\hat{y} \in \mathbb{R}^{d+1}}{\text{minimize}} \;\; Q(y - \hat{y}) \qquad \text{subject to} \qquad \operatorname{rk} H_r(\hat{y}) \leq r$$

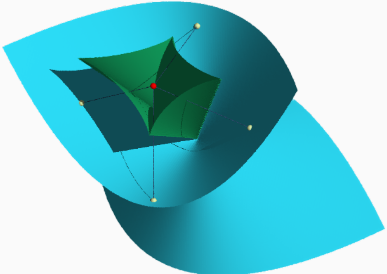
Euclidean Distance Degree

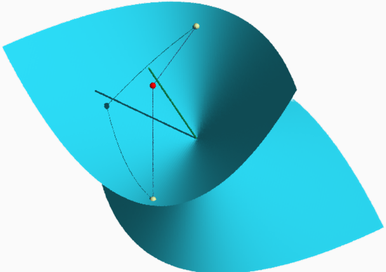
- Given a variety $X \subseteq \mathbb{C}^N$ and a point $y \in \mathbb{R}^N$, find closest point on $X(\mathbb{R})$
- Distance measured using non-degenerate real quadric $Q_\Lambda(x) := x^\intercal \Lambda x$

Definition (Euclidean distance degree, generic EDD)

The number of complex critical points of $\hat{y} \mapsto Q_\Lambda(\hat{y} - y)$ on X_{reg} for general $y \in \mathbb{R}^N$ is the Euclidean Distance degree $\mathrm{EDD}_\Lambda(X)$.

- If Λ is chosen generically, then obtain largest possible $\mathrm{EDD}_{\mathrm{gen}}(X)$



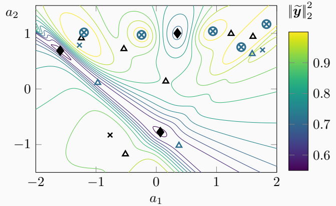


Secant varieties of the rational normal curve

- Rational normal curve $\mathbb{P}\mathcal{X}_{d,1} = \nu_d\mathbb{P}^1 = \mathrm{Im}(x \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d)) \subseteq \mathbb{P}^d$
- Secant variety $\sigma_r X = \overline{\bigcup_{p_1,\ldots,p_r \in X}\langle p_1,\ldots,p_r\rangle} \subseteq \mathbb{P}^d$, together

$$\mathbb{P}\mathcal{X}_{d,r} := \sigma_r\nu_d\mathbb{P}^1 = \left\{ y \in \mathbb{P}^d \;\middle|\; \mathrm{rank}\begin{bmatrix} y_0 & y_1 & \cdots & y_r \\ y_1 & y_2 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{d-r} & \hat{y}_{N-r} & \cdots & y_d \end{bmatrix} < r \right\} \subseteq \mathbb{P}^d$$

- $\mathcal{X}_{d,r} \subseteq \mathrm{Sym}^d\,\mathbb{C}^2$ is the variety of binary forms of (border)rank at most r
- Interesting quadrics: Unit $\mathbf{1}_{d+1}$, Bombieri $\Theta := \mathrm{diag}(\binom{d}{i})_{i=0}^d, \ldots$
- Previous work on $\mathrm{EDD}_Q(X_{d,r})$
  - [Ottaviani–Spaenlehauer–Sturmfels] generic Λ
  - [Sefat Panah] Bombieri norm Θ, $r = d/2$
  - [Kozhasov–Muniz–Qi–Sodomaco] all Λ, $r = 1$



Main results

Theorem ([K.–Lagauw])

1. *Give simple polynomial system whose roots correspond to critical points, "optimally" solvable using parameter continuation algorithms.*
2. *Explicitly construct determinantal scheme $\mathcal{B}_\Lambda \subseteq \mathbb{P}^r$ such that if $\mathcal{B}_\Lambda = \emptyset$, then $\mathrm{EDD}_\Lambda(\mathcal{X}_{d,r}) = \mathrm{EDD}_{\mathrm{gen}}(\mathcal{X}_{d,r})$.*
3. *Obtain new expression via Thom–Porteous formula*

$$\mathrm{EDD}_{\mathrm{gen}}(\mathcal{X}_{d,r}) = \sum_{j=1}^{r} \binom{d-2r+j}{j} 3^j = \sum_{i=1}^{r} \tbinom{d-r+1}{r-i} \tbinom{d-2r+i}{i} 2^i.$$

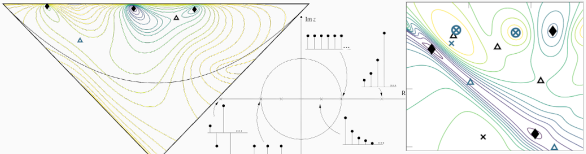
4. *If $\mathcal{B}_\Lambda$ is non-empty but finite, then $\deg \mathcal{B}_\Lambda \leq \mathrm{EDD}_{\mathrm{gen}}(\mathcal{X}_{d,r}) - \mathrm{EDD}_\Lambda(\mathcal{X}_{d,r})$, can computationally certify if equality holds.*

$\mathrm{EDD}_\Lambda(X_{d,r})$ for special quadrics

	$\Lambda = \mathbf{1}$ Unit				$\Lambda = F$ Frobenius				$\Lambda = \Theta$ Bombieri			
$d \setminus r$	1	2	3	4	1	2	3	4	1	2	3	4
1	1				1				1			
2	4				2				2			
3	7	1			7	1			3	1		
4	10	13			6	9			4	7		
5	13	34	1		13	34	1		5	16	1	
6	16	64	40		10	38	34		6	28	20	
7	19	103	142	1	19	103	142	1	7	43..45	62..64	1
8	22	151	334	121	14	103	246	113	8	61..65	134..142	53
9	25	208	643	547	25	208	643	543	9	82..88	243..263	(229)

- Method recovers *all* previously known values for $\mathrm{EDD}_\Theta(\mathcal{X}_{d,r})$, additionally

$$\mathrm{EDD}_\Theta(\mathcal{X}_{5,2}) = 16, \qquad \mathrm{EDD}_\Theta(\mathcal{X}_{6,2}) = 28$$



$\mathrm{EDD}_\Lambda(X_{d,r})$

$\mathrm{hf}_{I(X)_{\langle d\rangle}}(t)$

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Symmetric tensor decomposition

Definition (Waring decomposition, rank)

Let $F \in T_D = \mathbb{C}[X_0, \ldots, X_n]$. A *Waring decomposition* is an additive decomposition of the form $F = \sum_{i=1}^r L_i^D$, $L_i \in T_1$. The smallest possible r is the *Waring rank* of F.

- [Alexander–Hirschowitz]: The generic rank is (usually) $r = \lceil \frac{1}{n+1}\binom{n+D}{n} \rceil$
- In applications, symmetric tensors are often structured $\rightsquigarrow$ of subgeneric rank

Theorem ([Ballico], [Mella], [Chiantini–Ottaviani–Vannieuwenhoven], . . .)

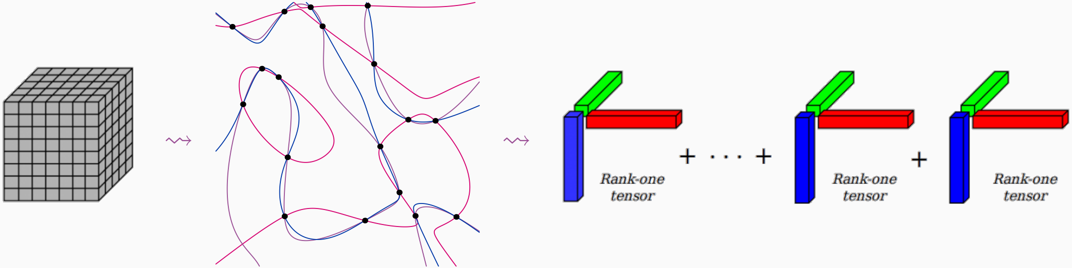
For $r(n+1) < \binom{n+D}{n}$, a general form of rank r is identifiable (unique minimal decomposition) except in the cases $(D, n, r) = (2, \geq 2, \geq 2), (6, 2, 9), (4, 3, 8), (3, 5, 9)$.

Running example:

A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ has $\operatorname{rk} F = \frac{1}{3}\binom{2+10}{2} = 22$. The set of such forms of rank 18 has dimension 54 in $\mathbb{C}^{66}$. Given F, how do we find the L_i algorithmically?

The catalecticant method

- ▷ Fix general $F = \sum_{i=1}^{r} L_i^D \in T_D$ of rank r
- ▷ Linear forms as points in projective space $[L_i] \in \mathbb{P}(T_1) \cong \mathbb{P}^n(\mathbb{C})$
- ▷ Catalecticant method give polynomials vanishing on $Z = \{[L_1], \ldots, [L_r]\} \subseteq \mathbb{P}^n$



$\rightsquigarrow$ Hope: Solutions to equations are exactly the $[L_i]$!

The algorithm

- Obtain *all* homog. equations of degree $\leq D/2$ vanishing on Z via kernel of catalecticant maps

$$\mathrm{Cat}_j(F)\colon \mathbb{C}[x_0,\dots,x_n]_j \to T_{D-j}, \qquad g \mapsto g(\partial_{X_0},\dots,\partial_{X_n})F(X_0,\dots,X_n)$$

- Algorithmic approach:
  1. Compute kernel basis $\mathcal{F}$ of the *linear* catalecticant map $\mathrm{Cat}_{\lfloor D/2 \rfloor}(F)$
  2. Solve *polynomial* system $\{\mathcal{F}=0\}$ to get $\mathbb{V}(\mathcal{F}) \stackrel{?}{=} \{[L_1],\dots,[L_r]\}$,
  3. Solve *linear* equations to get c_i in $F = \sum_{i=1}^r c_i L_i^D$
- (At least) three common approaches:
  - Gröbner bases computation (symbolic)
  - Homotopy continuation (numerical)
  - Eigenvalue/normal form methods (numerical/mixed)

$\rightsquigarrow$ Focus on the eigenvalue method approach here

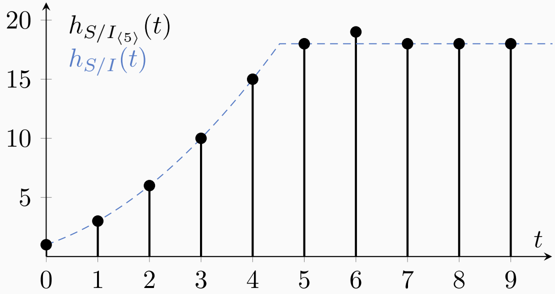
Eigenvalue/normal form methods for polynomial system solving

Task: Given 0-dim'l polynomial system $J \subseteq S$, compute $Z = \{z_1, \ldots, z_r\} = \mathbb{V}(J) \subseteq \mathbb{P}^n$

- For t large enough, $\mathrm{hf}_{S/J}(t) := \dim_{\mathbb{C}}(S/J)_t = r$ and $J_t = I(Z)_t$
- Multiplication map for $g \in S_e$: $M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/J)_{d+e}$
- Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has eigenvalues $\frac{g}{h}(z_i)$

$\rightsquigarrow$ Translate problem into large eigenvalue problem, solve numerically

- For this need $\mathrm{hf}_{S/J}(d+e) = \mathrm{hf}_{S/J}(d) = r$, want $d, d+e$ as small as possible



Rediscovering a notion introduced by [Ahmed–Fröberg–Rafiq]

Definition (Chopped ideal)

The *chopped ideal* of a homogeneous ideal $I \subseteq S$ in degree d is $I_{\langle d \rangle} := \langle I_d \rangle_S$.

From now on $Z \subseteq \mathbb{P}^n$ is a general set of r points, $I = I(Z)$

- The minimal generators of I live in degrees $\{d, d+1\}$, $d = \min\{\, t \mid \binom{n+t}{n} \geq r \,\}$.
- Can we recover Z from $I(Z)_{\langle d \rangle}$? When is $\mathbb{V}(I(Z)_d) = Z$?
- When does $(I(Z)_{\langle d \rangle})_{d+e} = I(Z)_{d+e}$? What is the Hilbert function $\mathrm{hf}_{I(Z)_{\langle d \rangle}}(t)$?

Lemma (Recovery for low-rank)

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$.

1. *If $r > \binom{n+d}{n} - n$, then $\mathbb{V}(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
2. *If $r = \binom{n+d}{n} - n$, then $\mathbb{V}(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
3. *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z scheme-theoretically.*

Example: $Z = 18$ points in the plane

t	$\dots$	3	4	5	6	7
$h_S(t)$	$\dots$	10	15	21	28	36
$h_I(t)$	$\dots$	0	0	3	10	18
$h_{I_{\langle 5\rangle}}(t)$	$\dots$	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$h_S(t)$	1	3	6	10	15	21	28	36
$h_{S/I}(t)$	1	3	6	10	15	18	18	18
$h_{S/I_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	19	18

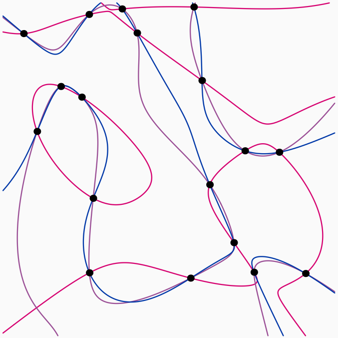
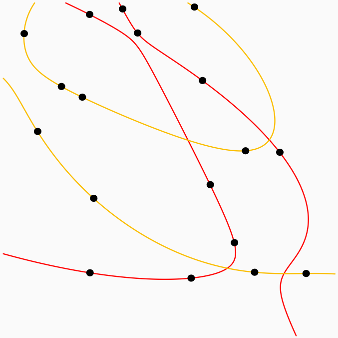
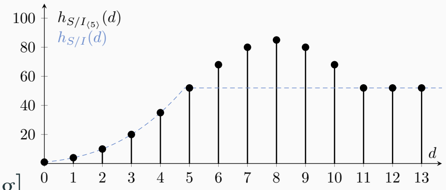


Figure 1: Three quintics $\langle q_1, q_2, q_3\rangle_{\mathbb{C}} = I_5$ passing through 18 general points (left) and the missing split sextic $cc' \in I_6$ (right).



The main conjecture

- ▷ If I_d has s generators, then $\binom{s}{2}$ Koszul syzygies $f_i \otimes f_j - f_j \otimes f_i$ appear in degree $2d$
- ▷ Assuming that generators have only the simplest possible higher syzygies suggests alternating sum
- ▷ This is lexicographic lower bound due to [Fröberg]



Expected syzygy conjecture (ESC)

If I is an ideal of r general points in $\mathbb{P}^n$, $d = \min\{\, t \mid \binom{n+t}{n} \geq r \,\}$ and $s = \mathrm{hf}_I(d)$, then

$$h_{S/I_{\langle d\rangle}}(t) = \begin{cases} \sum_{k\geq 0} (-1)^k \binom{s}{k} \cdot \mathrm{hf}_S(t-kd) & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the least integer $> d$ such that the sum is at most r.

Main results

Theorem ([Gesmundo–K.–Telen])

Conjecture (ESC) is true in the following cases:

- $r_{\max} := \binom{n+d}{n} - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- *In a large number of individual cases in low dimension (*`Macaulay2` *computation).*

The length of the saturation gap is bounded above by

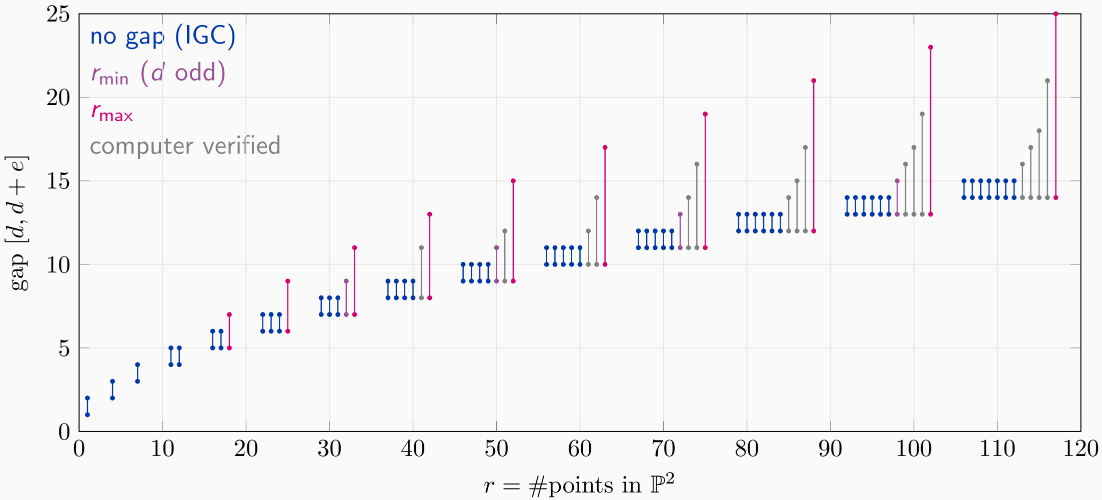
$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

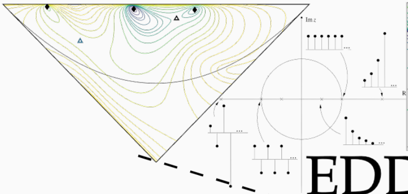
Whenever $I_{\langle d\rangle}$ *is non-saturated, one has* $\operatorname{reg}_{\mathrm{CM}} S/I_{\langle d\rangle} = \operatorname{reg}_{\mathrm{H}} S/I_{\langle d\rangle} - 1 = d + e - 1$.

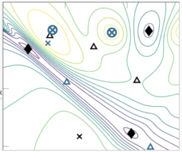
Tools: Syzygies, higher-codimension liaison, local cohomology

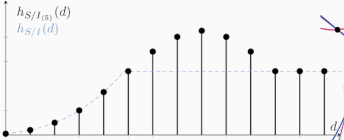
Visualization of the saturation gaps in $\mathbb{P}^2$

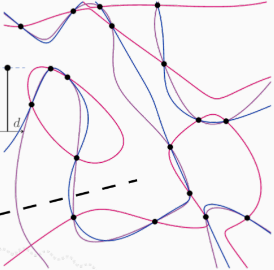
- ESC predicts exactly in which degrees we have $I_{\langle d \rangle} \subsetneq I$, the saturation gap











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Self-dual points

Let $\Gamma \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ be a set of $2n$ non-degenerate points identified with $\Gamma \in \mathbb{C}^{n\times 2n}$. **TFAE:**

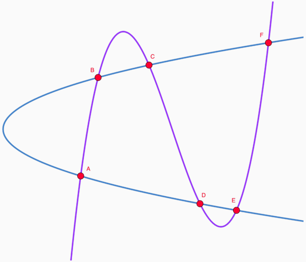
1. $\exists \Lambda \in \mathrm{Diag}(2n)$ invertible such that $\Gamma \cdot \Lambda \cdot \Gamma^{\mathsf{T}} = \mathbf{0}$ (fixed under *Gale transform*);
2. Subsets of $2n-1$ points impose the same number of conditions on quadrics as Γ
3. $\exists Q \in \mathrm{Sym}(n)$ non-deg. and $\Gamma = \Gamma_1 \,\dot\cup\, \Gamma_2$ s.t. Γ_1, Γ_2 are orthogonal bases w.r.t. Q:

$$\Gamma_i^{\mathsf{T}} \cdot Q \cdot \Gamma_i \in \mathrm{Diag}(n), \qquad i = 1, 2.$$

If Γ fails to impose independent cond's on quadrics by 1:

4. [Eisenbud–Popescu] The homogeneous coordinate ring $S_\Gamma = \mathbb{C}[\underline{x}]/I(\Gamma)$ is Gorenstein.

$\rightsquigarrow$ Slices of canonical curves are self-dual!



A parametrization of the moduli space

Let $\mathcal{A}_{n-1} \subseteq (\mathbb{P}^{n-1})^{2n} /\!/ \mathrm{SL}_n \times \mathfrak{S}_{2n}$ be the moduli space of self-dual points

1. All sets of four points in $\mathbb{P}^1$ are self-dual
2. Six points in $\mathbb{P}^2$ are self-dual iff intersection of quadric and cubic
3. A general set in $\mathcal{A}_3$ is a complete intersection of three quadric surfaces in $\mathbb{P}^3$
4. ... in $\mathcal{A}_4$ is a section of $X_6 = \mathrm{Gr}(2, \mathbb{C}^5) \subseteq \mathbb{P}^9$ with a quadric and a linear space
5. ... in $\mathcal{A}_5$ is a linear section of $X_7 = \mathrm{LG}_+(5, \mathbb{C}^{10}) \subseteq \mathbb{P}^{15}$
6. ... in $\mathcal{A}_6$ is a linear section of the Grassmannian $X_8 = \mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{P}^{14}$

1.–4. classical/[Eisenbud–Popescu], 5.–6. [Petrakiev], fails for $\mathcal{A}_7$

The Mukai slicing and lifting problem

$X_8 = \mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{P}^{14}$, $\operatorname{codim} X_8 = 6$, $\deg X_8 = 14$

- Slicing: Given a linear space $\mathbb{L} \subseteq \mathbb{P}^{14}$, compute the self-dual point configuration

$$\Gamma = \mathbb{L} \cap \mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{L} \cong \mathbb{P}^6$$

- Lifting: Given a self-dual points $\Gamma \subseteq \mathbb{P}^6$, find a $\mathbb{L} \in \mathbb{G}(6, \mathbb{P}^{14})$ and $L \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L}$

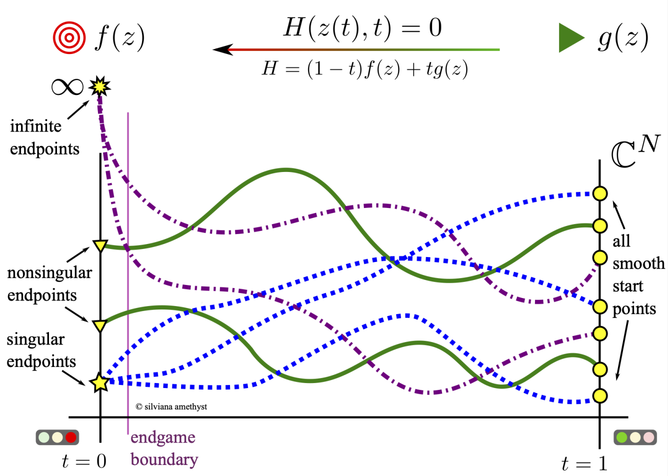
$$\Gamma = L^{-1}(\mathbb{L} \cap \mathrm{Gr}(2, \mathbb{C}^6))$$

- Numerical or symbolic? Complex or real?
- Also interesting for other X_g, or for canon. curves, K3 surfaces, Fano 3-folds, ...
- Computational problem posed by [Geiger–Hashimoto–Sturmfels–Vlad]

The wonders of numerical parameter continuation

- ▷ Slicing: Move linear space through Grassmannian
- ⇝ track intersection points

- ▷ Lifting: Move point configuration through $\mathcal{A}_6$ (?)
- ⇝ track some linear space in fiber of “slicing map”



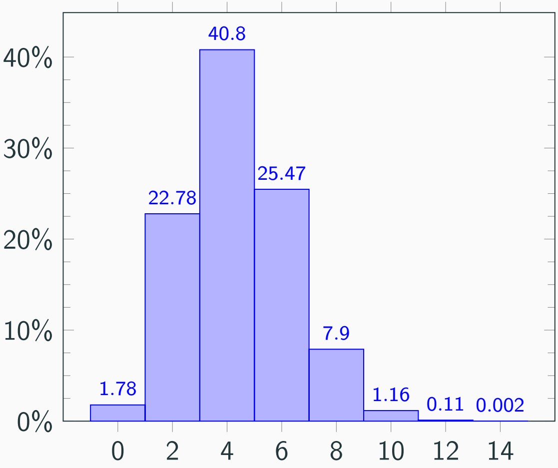
Slicing $X_8 = \mathrm{Gr}(2, \mathbb{K}^6) \subseteq \mathbb{P}^14$ using toric degeneration

Given $\mathbb{L} = \mathrm{Ker}\, A \subseteq \mathbb{P}^{14}$, find $\mathbb{L} \cap X_8$

- ▷ Toric degeneration of $\mathrm{Gr}(2, \mathbb{C}^6)$ via SAGBI basis

1. Solve for random $\mathbb{L}_0$ on toric variety (polyhedral start system)
2. Track via toric degeneration to X_8
3. Track $\mathbb{L}_0 \to \mathbb{L}$ (straight line homotopy)

Application: Slice $\mathrm{Gr}(2, \mathbb{R}^6) \subseteq \mathbb{P}^{14}(\mathbb{R})$ with 10,000,000 $\mathbb{L} \in \mathbb{G}(6, \mathbb{P}^{14}(\mathbb{R}))$ sampled uniformly, count real solutions



Main theoretical insight: A simple parametrization of $\mathcal{A}_n$

- $\mathcal{A}_n$ known to be rational variety [Dolgachev & Ortland]
- Orthogonal normal form: Non-degenerate self-dual points have representation

$$\Gamma = [I_n \mid P], \qquad P \in \mathrm{SO}(n, \mathbb{C})$$

- Skew normal form: Most self-dual points have (non-unique) representation by $S \in \mathrm{Skew}(n)$

$$\Gamma = [I_n + S \mid I_n - S] = \left[\begin{array}{cccc|cccc} 1 & s_1 & \cdots & s_{n-1} & 1 & -s_1 & \cdots & -s_{n-1} \\ -s_1 & 1 & \ddots & \vdots & s_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_{\binom{n}{2}} & \vdots & \ddots & \ddots & -s_{\binom{n}{2}} \\ -s_{n-1} & \cdots & -s_{\binom{n}{2}} & 1 & s_{n-1} & \cdots & s_{\binom{n}{2}} & 1 \end{array}\right]$$

- Use SNF this to move through $\mathcal{A}_6$ and track the linear slice Λ

A buffet of mathematical software

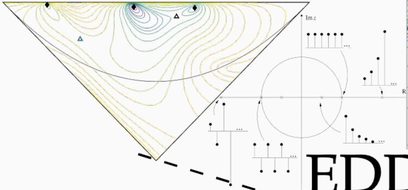
julia

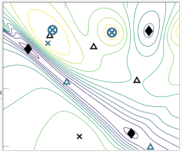
Homotopy Continuation.jl

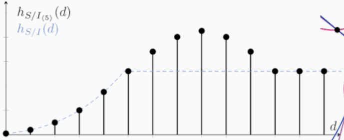
OSCAR
SYMBOLIC TOOLS

MATHREPO
MATHEMATICAL RESEARCH-DATA
REPOSITORY

Macaulay2 

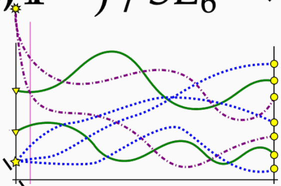


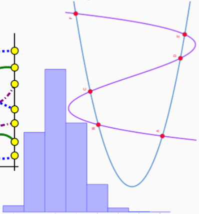




$\mathrm{EDD}_{\Lambda}(X_{d,r})$ $\mathrm{hf}_{I(X)_{\langle d\rangle}}(t)$

$\deg \nabla_{\log}(\mathcal{A})$ $\mathrm{Gr}\left(6,\mathbb{P}^{14}\right)/\,SL_6 \dashrightarrow \mathcal{A}_6$

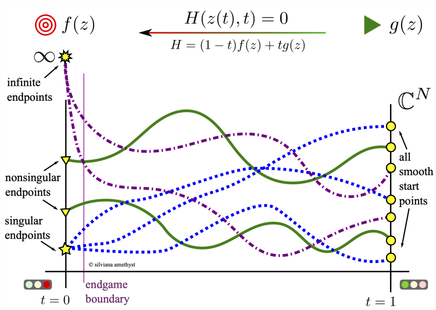


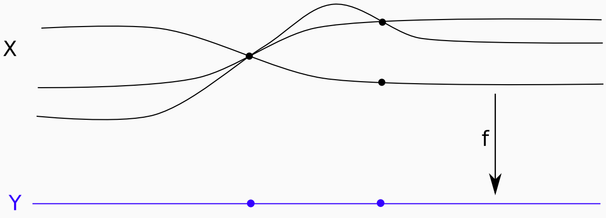


$\mathit{AlgMem}_{\mathbb{Q}} \in$ EXPSPACE

When parameter contiuation goes wrong

- Parameter continuation works well when solution paths are fare from each other
- When solutions come close, path jumping can appear
- The discriminant is the locus of parameters where solutions merge
- High school discriminant: $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$



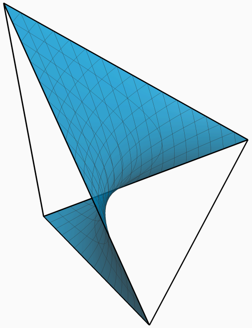


Maximum Likelihood Estimation in Algebraic Statistics

- Let $X \subseteq (\mathbb{C}^\times)^{n+1}$ be a d-dimensional smooth variety
- Discrete statistical model $X \cap \Delta_n = \{\, p \in X \cap \mathbb{R}^{n+1} \mid p_i > 0,\ p_0 + \cdots + p_n = 1 \,\}$
- Given data points $u \in \mathbb{N}^{n+1}$, which parameter maximizes the log-likelihood function

$$\mathcal{L}_u(x) = \log x_0^{u_0} \cdots x_n^{u_n}, \qquad x \in X \cap \Delta_n?$$

- Critical equations: $x \in \mathrm{Crit}_X(u) := \{\, x \in X \mid \nabla \mathcal{L}_u(x) = 0 \,\}$
- $\mathrm{Crit}_X(u)$ is a *finite* set of $\mathrm{MLdeg}(X)$ non-degenerate critical points for *general* data $u \in \mathbb{N}^{n+1}$ (or $\mathbb{C}^{n+1}$)
- [Huh] $\mathrm{MLdeg}(X) = |\chi(X)|$
- Extensively studied for toric models (exponential families), linear models, determinantal varieties, ...



Linear models and scattering amplitudes

- Let $\mathcal{A}$ be an essential arrangement of $n+1$ hyperplanes in $\mathbb{C}^d$

$$\mathcal{A} = \mathbb{V}(\ell_0) \cup \cdots \cup \mathbb{V}(\ell_n) \subseteq \mathbb{C}^d, \qquad (\ell_0(x), \ldots, \ell_n(x))^\mathsf{T} = Ax + b, \quad L^\mathsf{T} = [b \mid A]$$

- Parametrizes linear model $X := \mathbb{C}^d \setminus \mathcal{A} \xhookrightarrow{\ell} (\mathbb{C}^\times)^{n+1}$, can assume $\sum_j \ell_j = 1$
- Log-likelihood function or master function given by

$$\mathcal{L}_u(x) = u_0 \log \ell_0(x) + \cdots + u_n \log \ell_n(x), \qquad \nabla \mathcal{L}_u(x) = A^\mathsf{T} \operatorname{diag}(1/\ell_0, \ldots, 1/\ell_n) u$$

- Varchenko: For real arrangements, $\mathrm{MLdeg}(X) = \#$bounded chambers of $\mathcal{A} \cap \mathbb{R}^d$
- Critical equations appear as scattering equations in bi-adjoint scalar ϕ^3-theories [Cachazo–He–Yuan]

What is "general" data?

- Moving from general to special $u \in \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$, what can happen to $\mathrm{Crit}_X(u)$?
  1. Two critical points collide to form a non-reduced/degenerate point
  2. A positive-dimensional component appears
  3. A critical point disappears to infinity
- Outside of 1.-3. the finite set of critical points has constant size $\mathrm{MLdeg}(X)$
- The closure of 1.-3. was called the *data discriminant* in [Rodriguez–Tang]
- 3. was studied in [Sattelberger–van der Veer]

Definition (Logarithmic discriminant $\nabla_{\log}(X)$)

The *logarithmic discriminant* of a (smooth) variety $X \hookrightarrow (\mathbb{C}^\times)^{n+1}$ is

$$\nabla_{\log}(X) := \overline{\{\, u \in \mathbb{P}^n \mid \mathrm{Crit}_X(u) \text{ is infinite or non-reduced} \,\}}.$$

$\rightsquigarrow$ Goal: Understand logarithmic discriminants of hyperplane arrangements!

Three points enter a bar

- Three points on a line $\mathcal{A} = \{0, 1, b\} \subseteq \mathbb{C}^1$
- Model is a line $X = \mathbb{C}^1 \setminus \mathcal{A} \hookrightarrow (\mathbb{C}^\times)^3$ parametrized by $(x, x-1, x-b)$,

$$\mathcal{L}_u(x) = u_0 \log x + u_1 \log(x-1) + u_2 \log(x-b)$$

- Single critical equation in $x \in \mathbb{C}^1 \setminus \mathcal{A}$

$$\frac{u_0}{x} + \frac{u_1}{x-1} + \frac{u_2}{x-b} = 0 \quad \Longleftrightarrow \quad u_0(x-1)(x-b) + u_1 x(x-b) + u_2 x(x-1) = 0$$

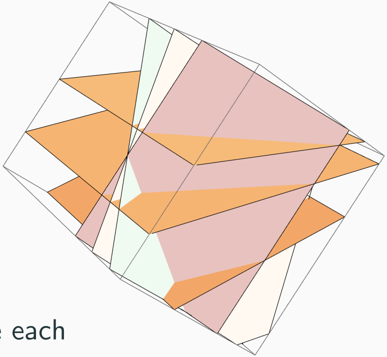
- When does this quadric in x have a double root? Highschool discriminant vanishes!

$$\Delta_{\log}(X) = (b-1)^2\, u_0^2 + 2b(b-1)\, u_0 u_1 + b^2\, u_1^2 - 2(b-1)\, u_0 u_2 + 2b\, u_1 u_2 + u_2^2$$

A split discriminant!

- Consider the arrangement $\mathcal{A}$ of six planes

$$\ell = (1, x_1, x_2, x_3) \cdot \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 0 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$



- The first and the last three planes intersect in a line each
- The logarithmic discriminant decomposes as

$$\begin{aligned} \nabla_{\log}(X) = {} & \mathbb{V}(144u_0^2 + 120u_0u_1 + 168u_0u_2 + 25u_1^2 - 70u_1u_2 + 49u_2^2) \\ & \cup \mathbb{V}(u_3^2 - 2u_3u_4 + 4u_3u_5 + u_4^2 + 4u_4u_5 + 4u_5^2) \\ & \cup \mathbb{V}(u_0 + u_1 + u_2, u_3 + u_4 + u_5). \end{aligned}$$

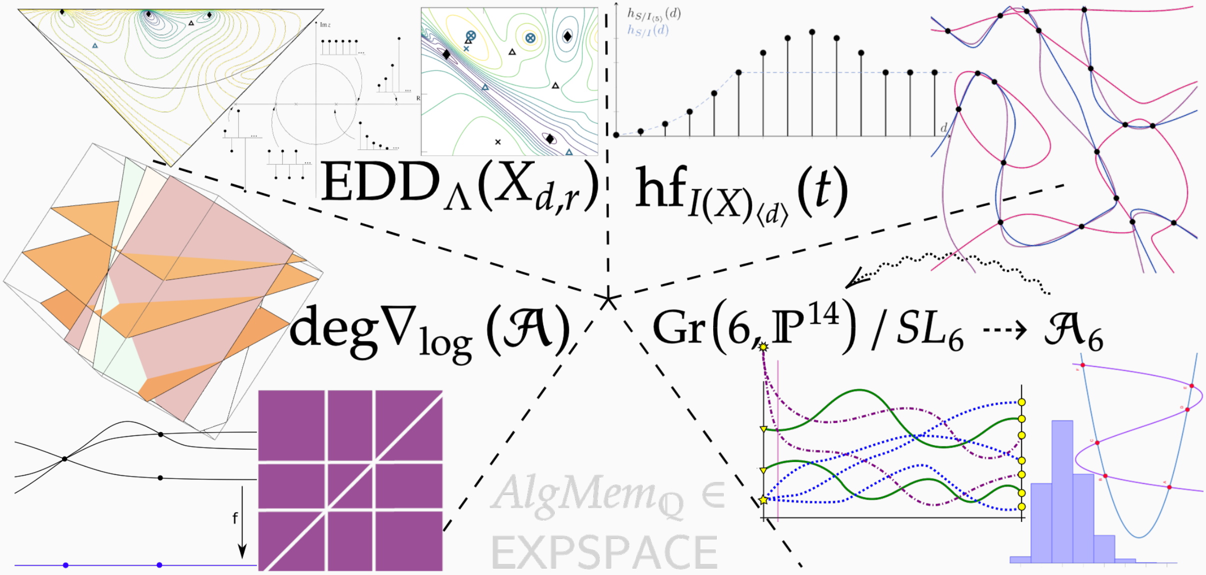
The Hurwitz Discriminant and general arrangements

Theorem ([K.–Kretschmer–Telen])

Let $\mathcal{A}$ be a bi-uniform arrangement of $n+1 \geq d+2$ hyperplanes in $\mathbb{C}^d$.

1. $\nabla_{\log}(X)$ *is an irreducible hypersurface.*
2. $\nabla_{\log}(X)$ *can be defined by a polynomial of degree* $2d\binom{n-1}{d}$
3. *If the arrangement is defined by real affine linear forms, then* $\nabla_{\log} \cap \mathbb{R}_{+}^{n+1} = \emptyset$.

- Main tool: Hurwitz discriminant $\nabla_{\mathrm{Hu}}(X) \supseteq \nabla_{\log}(X)$
  1. Reciprocal linear space $\mathcal{R} := \overline{\mathrm{Im}(\ell_0^{-1} : \cdots : \ell_n^{-1})} \subseteq \mathbb{P}^n$
  2. Hurwitz form $\mathcal{Z}_1(\mathcal{R}) \subseteq \mathrm{Gr}(n-d, \mathbb{P}^n)$, $\deg \mathcal{Z}_1(\mathcal{R}) = 2(n-d)\binom{n}{d-1}$
  3. $\nabla_{\mathrm{Hu}} := \varphi^{-1}(\mathcal{Z}_1(\mathcal{R}))$, pullback along $\varphi \colon \mathbb{P}^n \dashrightarrow \mathrm{Gr}(n-d, \mathbb{P}^n)$, $u \mapsto \mathrm{Ker}(A^{\mathsf{T}} \operatorname{diag}(u))$
- Key idea: $x \in \mathrm{Crit}_X(u)$ if and only if $(\ell_0^{-1}(x) : \cdots : \ell_n^{-1}(x)) \in \varphi(u) \cap \mathcal{R}$



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$AlgMem_{\mathbb{Q}} \in$ EXPSPACE
$h_{S/I_{\langle 5 \rangle}}(d)$
$h_{S/I}(d)$
f

Mathematicians always have problems

Definition (Computational problem, Decision problem)

A computational problem consists of an input, e.g. a tuple of data, and a question or expected output. A decision problem has output `yes` or `no`.

- ▷ Input/output encoded over finite alphabet Σ, $\Sigma^* := \{\text{words over } \Sigma\}$
- ▷ Decision problems are subsets $A \subseteq \Sigma^*$ (the "`yes`"-instances)

Definition (Ideal membership problem *IdealMem*$_{\mathbb{K}}$)

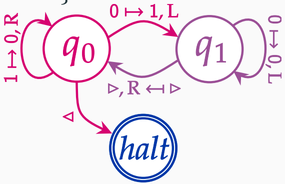
Input: $f_1, \dots, f_s, g \in R := \mathbb{K}[x_1, \dots, x_n]$

Question: $g \in \langle f_1, \dots, f_s \rangle_R$? (Decision problem)

Output: $h_1, \dots, h_s \in R$ with $g = h_1 f_1 + \cdots + h_s f_s$ (Representation problem)

Complexity classes 101

- ▷ Complexity of problem is measured with respect to model of computation
- ▷ Here use deterministic Turing machine (DTM), runtime $\approx$ #steps, memory $\approx$ #cells
- ▷ $\mathrm{TIME}(f) = \{$decision problem $A \mid \exists$DTM M deciding $w \in A$ in $O(f(|w|))$ steps$\}$
- ▷ $\mathrm{SPACE}(f) = \{A \mid \exists$DTM M deciding $w \in A$ using $O(f(|w|))$ cells$\}$



$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k) \overset{?}{\subseteq} \mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k)$$
$$\subseteq \mathrm{PSPACE} = \bigcup_k \mathrm{SPACE}(n^k) \subsetneq \mathrm{EXPSPACE} = \bigcup_k \mathrm{SPACE}(2^{n^k})$$

- ▷ $A \leq^{\mathrm{P}}_{\mathrm{m}} B$ if there is an efficiently computable map f with $w \in A$ iff $f(w) \in B$
- ▷ A is complete for a class $\mathcal{C}$ if $A \in \mathcal{C}$ and $A' \leq^{\mathrm{P}}_{\mathrm{m}} A$ for all $A' \in \mathcal{C}$

The (not so) ideal world

Theorem ([Hermann], [Mayr–Meyer], [Mayr])

1. *If* $g = h_1 f_1 + \cdots + h_s f_s$*, then* $\exists (h_i)_i$ *with* $\deg h_i \leq \deg g + (s \cdot \max_i \deg f_i)^{2^n}$.
2. *Degree bound* $O((sd)^{2^n})$ *for certificates* $(h_i)_i$ *is sharp.*
3. *IdealMem*$_{\mathbb{Q}}$ *is* EXPSPACE*-complete, even for binomial ideals.*
4. *The restriction IdealMem*$_{\mathbb{Q}}$(homog) *is* PSPACE*-complete,*

- Gröbner bases can still be doubly-exponential even for homogeneous ideals
- Deciding whether $1 \in \langle f_1, \ldots, f_s \rangle_R$ ("Nullstellensatz") is low in poly. hierarchy
- There are dimension-dependent degree bounds available [Mayr–Ritscher]
- The complexity of computing Gröbner bases seems to be linked to its Castelnuovo-Mumford regularity [Bayer–Mumford]

Subgalgebra Analogue to Membership Problem for Ideals

Definition (Subalgebra membership problem $AlgMem_{\mathbb{K}}$)

Input: $f_1, \ldots, f_s, g \in \mathbb{K}[x_1, \ldots, x_n]$

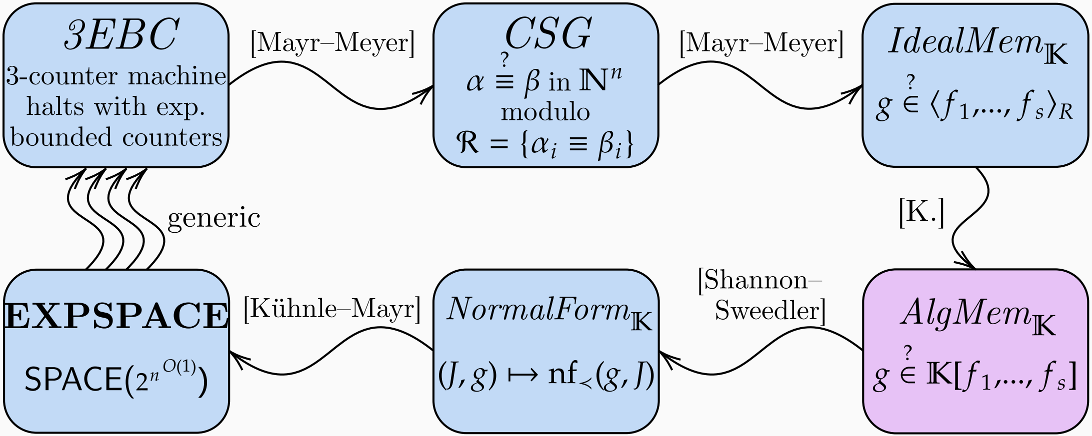
Question: $g \in \mathbb{K}[f_1, \ldots, f_s]$? (Decision problem)

Output: $p \in \mathbb{K}[t_1, \ldots, t_s]$ with $g = p(f_1, \ldots, f_s)$ (Certification problem)

Fundamental questions:

1. Degree bounds on p depending on $n, s, \deg f_i$?
2. Upper and lower bounds on complexity of $AlgMem_{\mathbb{Q}}$? Related to $IdealMem_{\mathbb{Q}}$?
3. Easier when the polynomials are homogeneous? Or monomials? Or n bounded?
4. The analogue to Gröbner bases for ideals are SAGBI bases for subalgebras. What is the complexity of SAGBI bases?

A chain of reductions



The upper bound

Theorem ([K.])

$AlgMem_{\mathbb{Q}}$ is in $\mathrm{EXPSPACE}$ and $AlgMem_{\mathbb{Q}}(\text{homog})$ is in PSPACE.
A certificate $p \in \mathbb{Q}[t_1, \ldots, t_s]$ can be computed using $2^{O(|w|)}$ working space.

Idea. Combine elimination method Shannon–Sweedler 1986 with the exponential working space algorithm for normal forms by [Kühnle & Mayr 1996]. □

- ▷ Careful analysis reveals that bounded variable case is also in PSPACE
- ▷ We also get a degree bound for the certificate using the Dubé bound:

Theorem ([K.])

If $g \in \mathbb{K}[f_1, \ldots, f_s]$, $e := \deg g$, then there is a p with $p(f_1, \ldots, f_s) = g$ of degree

$$\deg p \leq e + \left(\left(\tfrac{1}{2} d^{2s^2} + d \right)^{2^n} + 1 \right)^{(n+s)^2+1} e^{n+s} \approx d^{O((n+s)^4 2^n)} e^{n+s}.$$

The exponential space lower bound

Lemma

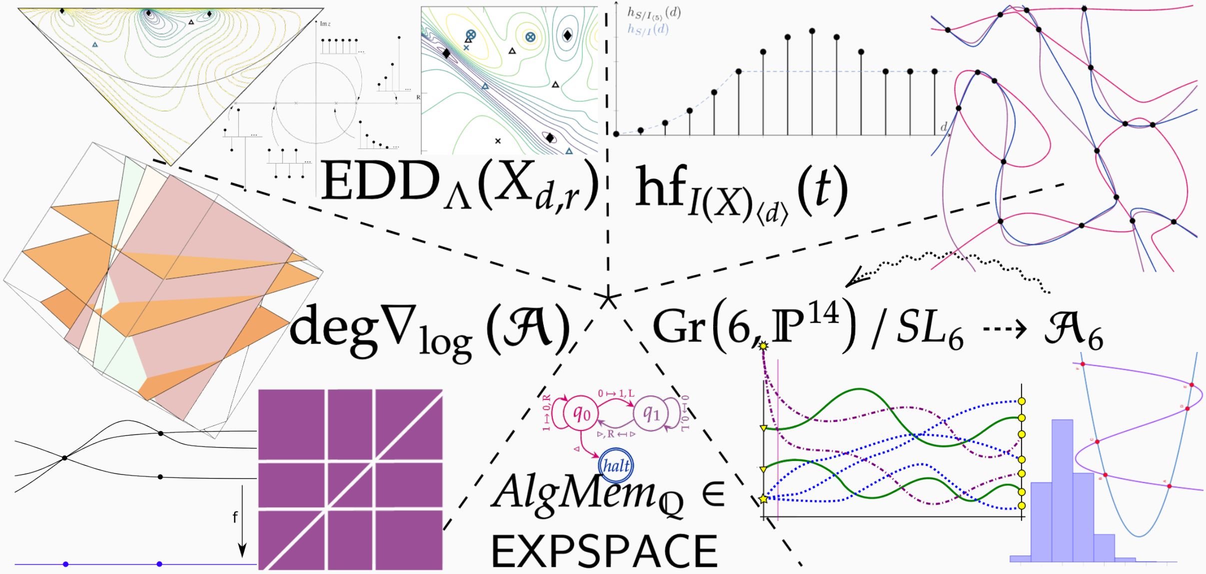
Let $f_1, \ldots, f_s, g \in R = \mathbb{K}[x_1, \ldots, x_n]$, then the following are equivalent:

1. $g \in \langle f_1, \ldots, f_s \rangle_R$;
2. $ug \in A := \mathbb{K}[x_1, \ldots, x_n, uf_1, \ldots, uf_s] \subseteq R[u]$.

The minimal degree of $p \in \mathbb{K}[t_1, \ldots, t_{n+s}]$ with $p(x_1, \ldots, uf_s) = ug$ is one less than the minimal degree of a representation $\max_i \deg h_i$. The minimal number of terms of p coincides with the minimal total number of terms of $h_1, \ldots, h_s$.

Theorem (K. 2025)

1. *IdealMem*$_{\mathbb{Q}} \leq_m^{\mathrm{P}}$ *AlgMem*$_{\mathbb{Q}}$, *thus AlgMem*$_{\mathbb{Q}}$ *is* $\mathrm{EXPSPACE}$*-complete.*
2. *Similar for homogeneous polynomials, AlgMem*$_{\mathbb{Q}}(\mathrm{homog})$ *is* PSPACE*-complete.*
3. *For monomial algebras, and hence for SAGBI bases, the problem is still* NP*-hard.*



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$h_{S/I}(d)$
d
Im z
q_0
q_1
$0 \mapsto 1, L$
halt
f

Mathematics is a team sport!

Thank you! Time for questions!

To infinity and beyond

- Identify $\mathcal{X}_{d,r} = \widehat{\sigma}_r \nu_d(\mathbb{P}^1)$ with sequences $y \in \mathbb{C}^{d+1}$ satisfying order r linear recurrence
- Fixing $r \in \mathbb{N}$ and increasing $d \to \infty$ leads in the inverse limit to the variety $\mathcal{X}_{\infty,r}$

$$\widehat{\sigma}_r \nu_7 \mathbb{P}^1 \twoheadleftarrow \widehat{\sigma}_r \nu_8 \mathbb{P}^1 \twoheadleftarrow \widehat{\sigma}_r \nu_9 \mathbb{P}^1 \twoheadleftarrow \cdots \twoheadleftarrow \widehat{\sigma}_r \nu_\infty \mathbb{P}^1 = \{\, y \in \mathbb{C}^{\mathbb{N}} \mid y \text{ obeys order } r \text{ lin. recurrence} \,\}$$

- Naturally parametrized by rational functions [Kronecker]

$$y \mapsto Y(z) = \frac{b(z)}{a(z)} = \sum_{j=1} y_j z^{-j} \in \mathbb{C}(z), \qquad y = (y_j)_j = \big(\textstyle\sum_{a(\lambda)=0} c_\lambda \lambda^j\big)_j$$

- $\|\cdot\|_2$ on $\mathbb{R}^{d+1}$ becomes ℓ^2-norm on sequences $\|y\|_{\ell^2}^2 = \|Y\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{S}^1} |Y(z)| \frac{\mathrm{d}z}{z}$

$\rightsquigarrow$ Impose Schur-stability of $a(z)$ ($|\lambda| < 1$)

- Approximation from $\mathcal{X}_{\infty,R}(\mathbb{R})$ to $\mathcal{X}_{\infty,r}(\mathbb{R})$ known as model order reduction

Walsh meets Porteous: ℓ^2-distance degree of $\mathcal{X}_{\infty,r}$

- Model order reduction problem: Given $Y(z) = \frac{d(z)}{c(z)}$ (c stable, $\deg d < \deg c \le R$)

$$\underset{\widehat{Y}}{\text{minimize}} \ \|\widehat{Y} - Y\|^2_{\mathcal{H}_2}, \qquad \text{subject to } \widehat{Y} = \frac{b(z)}{a(z)}, \ \deg b < \deg a \le r, \ a \text{ stable}$$

- [Walsh]: Criticality is characterized by $Y(\omega^{-1}) = \widehat{Y}(\omega^{-1})$, $Y'(\omega^{-1}) = \widehat{Y}'(\omega^{-1})$

Theorem ([Lagauw–K])

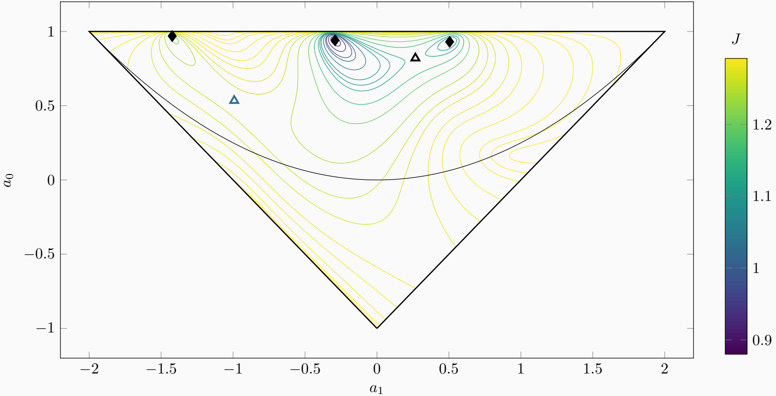
- *The critical incidence $\{(Y, \widehat{Y}) \mid \widehat{Y}$ critical point to $Y\}$ is an irreducible variety.*
- *The algebraic degree of model order reduction (order R to r) is*

$$\sum_{j=0}^{r} \binom{R - r - 1 + j}{j} 2^j.$$

Example: Model order reduction from order 6 to 2

$$Y(z) = \frac{0.0448z^5 + 0.2368z^4 + 0.0013z^3 + 0.0211z^2 + 0.2250z + 0.0219}{z^6 - 1.2024z^5 + 2.3675z^4 - 2.0039z^3 + 2.2337z^2 - 1.0420z + 0.8513}$$

$$y = (0.0448, 0.2907, 0.2447, 0.2830, 0.2123, 0.3245, 0.0438, 0.5013, 0.4671, \dots)$$



Mukai Grassmannians

Theorem (Shigeru Mukai (1987))

For a genus $g \geq 6$, a sufficiently general

- *canonical curve ($g \leq 9$),*
- *pseudo-polarized K3 surface ($g \leq 10$), or*
- *prime Fano 3-fold ($g \leq 10$, $g = 12$)*

is a linear[1] *section of a homogeneous variety $X_g \subseteq \mathbb{P}V$.*

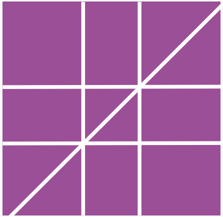
g	V	$X_g \subseteq \mathbb{P}V$	$\dim X_g$	$\dim \mathbb{P}V$
6	$\bigwedge^2 \mathbb{C}^5$	$\mathrm{Gr}(2, \mathbb{C}^5)$	6	9
7	$\bigwedge^{\text{even}} \mathbb{C}^{10}$	$\mathrm{LG}_+(5, \mathbb{C}^{10})$	10	15
8	$\bigwedge^2 \mathbb{C}^6$	$\mathrm{Gr}(2, \mathbb{C}^6)$	8	14
9	$\bigwedge^3 \mathbb{C}^6 / \omega \wedge \mathbb{C}^6$	$\mathrm{Gr}_\omega(3, \mathbb{C}^6)$	6	13

[1] For $g = 6$ additionally intersect with a quadric

The discriminant of $\mathcal{M}_{0,m}$

- $\mathcal{M}_{0,m}$ parametrizes tuples of m distinct points on $\mathbb{P}^1$
- Fixing $(0, 1, x_1, \ldots, x_{m-3}, \infty)$, it can be realized in $\mathbb{C}^{m-3}$ as the complement of the minors of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & x_{m-3} & 1 \end{bmatrix}$$



- Mandelstam variables s_{ij} corresponding to minor (i, j)
- Discriminant for $m = 5$ has degree $4 < 2 \cdot 2 \cdot \binom{5-2}{2} = 12$

$$\Delta_{\log}(\mathcal{M}_{0,5}) = (s_{13}s_{24} + s_{13}s_{34} + s_{14}s_{34} + s_{14}s_{23} + s_{23}s_{34} + s_{24}s_{34} + s_{34}^2)^2 - 4s_{13}s_{14}s_{23}s_{24}$$

- The Hurwitz discriminant has the extra factors

$$\Delta_{\mathrm{Hu}}(\mathcal{M}_{0,5}) = (s_{13} + s_{23} + s_{34})^2 \cdot (s_{14} + s_{24} + s_{34})^2 \cdot \Delta_{\log}(\mathcal{M}_{0,5})$$