Derived functors done quick

Doing exercise 11.2 the hard way

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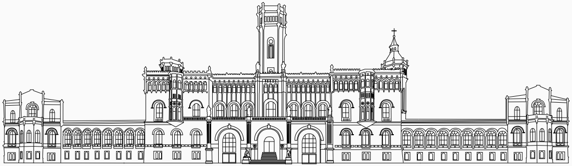


What exactly is going on?

Many functors $\mathcal{T}$ of abelian groups or, more generally, R-Modules preserve split-exact sequences, but do *not* preserve exactness in general.

- $A \mapsto \mathrm{Hom}(A, G)$ is contravariant & left-exact (Satz III.2.3)
  $A \mapsto \mathrm{Hom}(G, A)$ is covariant & left-exact
- For fixed M, $A \mapsto A \otimes M$ is covariant & right-exact
- $A \mapsto \mathrm{Tor}(A)$ (torsion subgroup) is covariant & left-exact
- M a G-module (G group), then taking *invariants* $M \mapsto M^G$ is covariant & left-exact

Defining the groups $R^i\mathcal{T}(A)$ via free resolution

Fix a contravariant left-exact functor $\mathcal{T}$ such as $\mathrm{Hom}(-, G)$.

- For a given abelian group A choose a *free resolution*

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \longrightarrow 0$$

- Apply the functor $\mathcal{T}$ to obtain a cochain complex

$$0 \longrightarrow \mathcal{T}(A) \xhookrightarrow{\mathcal{T}(d_0)} \mathcal{T}(F_0) \xrightarrow{\mathcal{T}(d_1)} \mathcal{T}(F_1) \xrightarrow{\mathcal{T}(d_2)} \mathcal{T}(F_2) \longrightarrow \cdots$$

- Take cohomology of the complex $C^\bullet \coloneqq (0 \to \mathcal{T}(F_0) \to \mathcal{T}(F_1) \to \mathcal{T}(F_2) \to \dots)$

$$(R^i\mathcal{T})(A) \coloneqq H^i(C^\bullet)$$

- Observe $R^0\mathcal{T}(A) = \ker \mathcal{T}(d_1) \cong \mathcal{T}(A)$ ($\mathcal{T}$ is left-exact!)

Let's turn $R^i\mathcal{T}$ into a $\mathcal{F}$unctor!

Let $g\colon A \to B$ be a homomorphism and fix free resolutions of A, B.

(i) One may extend g to a chain map $g_\bullet$ (map basis of F_j to suitable preimages)

$$\begin{array}{ccccccccc} \dots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \\ \dots & \longrightarrow & F_2 & \longrightarrow & F_0' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Apply $\mathcal{T}$ to obtain a cochain map $\mathcal{T}(g_\bullet)$ which induces homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}(F_0) & \longrightarrow & \mathcal{T}(F_1) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \mathcal{T}(g_0)} & & \downarrow{\scriptstyle \mathcal{T}(g_1)} & & \\ 0 & \longrightarrow & \mathcal{T}(F_0) & \longrightarrow & \mathcal{T}(F_1) & \longrightarrow & \dots \end{array} \quad \overset{\text{take}}{\underset{\text{cohomology}}{\Longrightarrow}} \quad R^i\mathcal{T}(g)\colon R^i\mathcal{T}(A) \to R^i\mathcal{T}(B)$$

(ii) If $(g_i')_i$ is another extension, then $\mathcal{T}(g_\bullet)$ and $\mathcal{T}(g_\bullet')$ are chain homotopic, in particular they induce the same map $R^i\mathcal{T}(A) \to R^i\mathcal{T}(B)$.

The case $\mathcal{T} = \operatorname{Hom}(-, G)$

- We used two term free resolutions $0 \to R \xrightarrow{i} F \to A \to 0$.

$$C^\bullet = (0 \to \operatorname{Hom}(A, G) \to \operatorname{Hom}(F, G) \to \operatorname{Hom}(R, G) \to 0)$$

- We can compare the Ext groups from the lecture to the derived functor of $\operatorname{Hom}(-, G)$:

$$\operatorname{Ext}(A, G) \coloneqq \operatorname{Hom}(R, G)/\operatorname{im}(i^\sharp) \eqqcolon H^1(C^\bullet) = \operatorname{Ext}^1(A, G)$$

  where $\operatorname{Ext}^i = R^i \operatorname{Hom}(-, G)$.
- Furthermore, as $C^i = 0$ for $i \geqslant 2$, we get $\operatorname{Ext}^i(A)$ for all $i \geqslant 2$ („no higher Ext“)

Hence we get a sequence $\operatorname{Ext}(C, G) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}(A, G)$.

$\rightsquigarrow$ Want to obtain a connecting homomorphism

Do snakes dream of long exact sequences?

- Intuition: The groups $R^i\mathcal{T}$ measure the failure of $\mathcal{T}$ to be exact.
- Concretely: Given a short exact sequence $0 \to A \to B \to C \to 0$ we want to extend the exact sequence

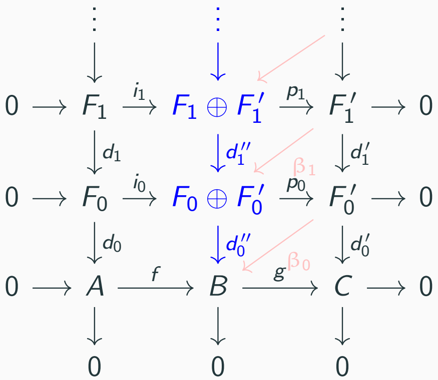
$$0 \to \mathcal{T}(C) \to \mathcal{T}(B) \to \mathcal{T}(A) \not\to 0$$

to a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{T}(C) \longrightarrow \mathcal{T}(B) \longrightarrow \mathcal{T}(A) \\ \hookrightarrow R^1\mathcal{T}(C) \longrightarrow R^1\mathcal{T}(B) \longrightarrow R^1\mathcal{T}(A) \\ \hookrightarrow R^2\mathcal{T}(C) \longrightarrow R^2\mathcal{T}(B) \longrightarrow R^2\mathcal{T}(A) \\ \hookrightarrow R^3\mathcal{T}(C) \longrightarrow \quad \dots \end{array}$$

The horseshoe lemma - simultaneous resolutions

Given free resolutions $(F_i)_i$ of A and $(F_i')_i$ of C, there is a free resolution of the form $(F_i \oplus F_i')_i$ such that the following diagram is commutative and exact:



- Define $\beta_0 \colon F_0' \to B$ by mapping a basis to preimages of $d_0'(b_i)$ under g
- Define $d_0'' \coloneqq f \circ d_0 \oplus \beta_0$
- Define $\beta_1 \colon F_1' \to F_0 \oplus F_0'$ by mapping a basis to preimages of $d_1'(b_i)$ under p_0
- Define $d_1'' \coloneqq i_0 \circ d_1 \oplus \beta_1$
- ...

This is a resolution: Apply the LES of homology to this exact sequence of complexes and use that the outer columns are exact

The long exact sequence on derived functors

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow d_2 & & \downarrow d_2'' & & \downarrow d_2' & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_1 \oplus F_1' & \longrightarrow & F_1' & \longrightarrow & 0 \\
 & & \downarrow d_1 & & \downarrow d_1'' & & \downarrow d_1' & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 \oplus F_0' & \longrightarrow & F_0' & \longrightarrow & 0 \\
 & & \downarrow d_0 & & \downarrow d_0'' & & \downarrow d_0' & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\quad \overset{\mathcal{T}(-)}{\Longrightarrow} \quad
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \mathcal{F}(d_2)\uparrow & & \mathcal{F}(d_2'')\uparrow & & \mathcal{F}(d_2')\uparrow & & \\
0 & \longleftarrow & \mathcal{F}(F_1) & \longleftarrow & \mathcal{F}(F_1 \oplus F_1') & \longleftarrow & \mathcal{F}(F_1') & \longleftarrow & 0 \\
 & & \mathcal{F}(d_1)\uparrow & & \mathcal{F}(d_1'')\uparrow & & \mathcal{F}(d_1')\uparrow & & \\
0 & \longleftarrow & \mathcal{F}(F_0) & \longleftarrow & \mathcal{F}(F_0 \oplus F_0') & \longleftarrow & \mathcal{F}(F_0') & \longleftarrow & 0 \\
 & & \mathcal{F}(d_0)\uparrow & & \mathcal{F}(d_0'')\uparrow & & \mathcal{F}(d_0')\uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The right diagram is a short exact sequence of complexes, as the rows are split-exact.
Apply LES (II.5.1) to obtain

$$0 \to \mathcal{T}(C) \to \mathcal{T}(B) \to \mathcal{T}(A) \to R^1\mathcal{T}(C) \to R^1\mathcal{T}(B) \to R^1\mathcal{T}(A) \to R^2\mathcal{T}(C) \to R^2\mathcal{T}(B) \to R^2\mathcal{T}$$

Interesting consequences

This solves the exercise!

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \longrightarrow \mathrm{Ext}(C,G) \longrightarrow \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G) \longrightarrow 0$$

In particular $\mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G)$ is surjective (ex. 11.3(i)).

This construction can be extended to other functors as mentioned in the beginning.

For a nice introduction see

https://r0hilp.github.io/assets/docs/tutorial_derived_functors.pdf