Specific Euclidean Distance Degrees of Secant Varieties to the Rational Normal Curve

From Geometry to Numbers: a celebration of women in mathematics

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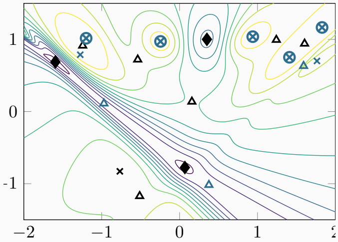

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Euclidean Distance Degree

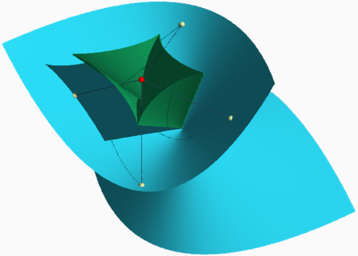
- Given a variety $X \subseteq \mathbb{C}^N$ and a point $y \in \mathbb{R}^N$, find closest point on $X(\mathbb{R})$
- Distance measured using non-degenerate (real) quadric $Q_\Lambda(x) := x^\intercal \Lambda x$

Definition ([Draisma–Horobeț–Ottaviani–Sturmfels–Thomas 2015])

The number of complex critical pts of $\hat{y} \mapsto Q_\Lambda(\hat{y} - y)$ on X_{reg} for general $y \in \mathbb{R}^N$ is the Euclidean Distance degree $\text{EDD}_\Lambda(X)$. $\text{EDD}_\Lambda(X \subseteq \mathbb{P}^{N-1}) := \text{EDD}_\Lambda(\widehat{X})$.

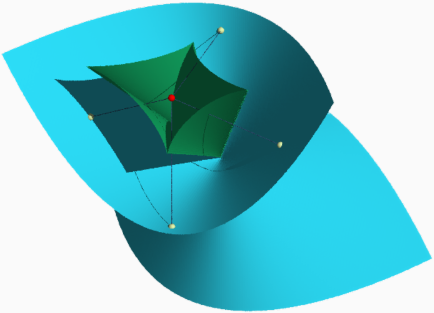
- Formally: Consider distance correspondence

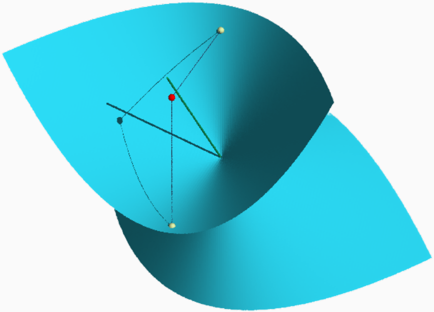
$$\mathcal{E}_X = \overline{\{\, (y, \hat{y}) \mid \hat{y} \in X_{\text{reg}},\ y - \hat{y} \perp_\Lambda T_{\hat{y}}X \,\}} \subseteq \mathbb{C}^N \times X$$



- Projection $\pi_1 \colon \mathcal{E}_X \to \mathbb{C}^N$ is generically finite, $\text{EDD}_\Lambda(X) := \deg \pi_1$ (or 0 if π_1 not dominant)
- If $\Lambda \succ 0$ and $X_{\text{reg}}(\mathbb{R}) \neq \emptyset$, then $\text{EDD}_\Lambda(X) > 0$

Generic vs specific EDD





- For general quadric obtain generic EDD, $\mathrm{EDD}_{\mathrm{gen}}(X) \geq \mathrm{EDD}_{\Lambda}(X)$ for all Λ
- $\mathrm{EDD}_{\mathrm{gen}}$ of $X \subseteq \mathbb{P}^d$ is determined by Chern(-Mather) classes $c_i(X)$

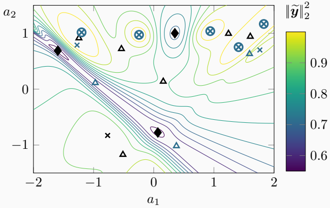
$$\mathrm{EDD}_{\mathrm{gen}}(X) = \sum_{i=0}^{\dim X} (-1)^i \cdot (2^{\dim X+1-i} - 1) \cdot \deg_{\mathbb{P}^d} c_i(X).$$

Secant varieties of the rational normal curve

- Rational normal curve $X_{d,1} = \nu_d \mathbb{P}^1 = \mathrm{Im}(x \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d)) \subseteq \mathbb{P}^d$
- Secant variety $\sigma_r X = \overline{\bigcup_{p_1,\ldots,p_r \in X} \langle p_1, \ldots, p_r \rangle} \subseteq \mathbb{P}^d$, together

$$X_{d,r} := \sigma_r \nu_d \mathbb{P}^1 = \left\{ y \in \mathbb{P}^d \;\middle|\; \mathrm{rank} \begin{bmatrix} y_0 & y_1 & \cdots & y_r \\ y_1 & y_2 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{d-r} & \hat{y}_{N-r} & \cdots & y_d \end{bmatrix} < r \right\} \subseteq \mathbb{P}^d$$

- $X_{d,r} \subseteq \mathbb{P}\,\mathrm{Sym}^d\,\mathbb{C}^2$ is the variety of binary forms of (border)rank at most r
- Distance optimization to $X_{d,r}(\mathbb{R})$ relevant in systems engineering
- Interesting quadrics: Unit $\mathbf{1}_{d+1}$, Bombieri $\Theta := \mathrm{diag}(\binom{d}{i})_{i=0}^d$, ...
- Previous work on $\mathrm{EDD}_Q(X_{d,r})$
  - [Ottaviani-Spaenlehauer-Sturmfels 2014] generic Λ
  - [Sefat Panah 2025] Bombieri norm Θ, $r = d/2$
  - [Kozhasov-Muniz-Qi-Sodomaco 2025] all Λ, $r = 1$



Main results

Theorem ([Kayser–Lagauw 2026+])

1. *Give simple polynomial system whose roots correspond to critical points, "optimally" solvable using parameter continuation algorithms.*
2. *Explicitly construct determinantal scheme $\mathcal{B}_\Lambda \subseteq \mathbb{P}^r$ such that if $\mathcal{B}_\Lambda = \emptyset$, then $\mathrm{EDD}_\Lambda(X_{d,r}) = \mathrm{EDD}_{\mathrm{gen}}(X_{d,r})$.*
3. *Obtain new expression via Thom–Porteous formula*

$$\mathrm{EDD}_{\mathrm{gen}}(X_{d,r}) = \sum_{j=1}^{r} \binom{d-2r+j}{j} 3^j = \sum_{i=1}^{r} \binom{d-r+1}{r-i}\binom{d-2r+i}{i} 2^i.$$

4. *If $\mathcal{B}_\Lambda$ is non-empty but finite, then $\deg \mathcal{B}_\Lambda \leq \mathrm{EDD}_{\mathrm{gen}}(X_{d,r}) - \mathrm{EDD}_\Lambda(X_{d,r})$, can computationally certify if equality holds.*

$\mathrm{EDD}_\Lambda(X_{d,r})$ for special quadrics

$d \setminus r$	$\Lambda = 1$ Unit 1	2	3	4	$\Lambda = F$ Frobenius 1	2	3	4	$\Lambda = \Theta$ Bombieri 1	2	3	4
1	1				1				1			
2	4				2				2			
3	7	1			7	1			3	1		
4	10	13			6	9			4	7		
5	13	34	1		13	34	1		5	16	1	
6	16	64	40		10	38	34		6	28	20	
7	19	103	142	1	19	103	142	1	7	43..45	62..64	1
8	22	151	334	121	14	103	246	113	8	61..65	134..142	53
9	25	208	643	547	25	208	643	543	9	82..88	243..263	229

- Method recovers *all* previously known values for $\mathrm{EDD}_\Theta(X_{d,r})$, additionally

$$\mathrm{EDD}_\Theta(X_{5,2}) = 16, \qquad \mathrm{EDD}_\Theta(X_{6,2}) = 28$$

Thank you! Questions?

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Image credit

- Slide 1, 2: Thanks to Luca Sodomaco for letting me use his graphics!
- Slide 0, 3: Contour plot by Sibren Lagauw using MatLab and pgfplots