Hilbert Functions of Chopped Ideals

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- 👋 2^{nd} year PhD student at MPI MiS w/ Simon Telen
- 🏫 Studied both math and CS at LUH (Hannover)
- 💙 Algebra in all flavors, Algebraic Geometry, Tensor Decomposition, Algorithms, Complexity Theory, ...
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- 🍓 **Fun fact:** I can beat the video game *Celeste* in <40min

Me, Fulvio & Simon

There's geometry hidden behind eigenvalue methods for symmetric tensor decomposition!

(symmetric) tensor decomposition

What is a tensor?

A tensor...

- ...is an object that transforms like a tensor
- ...is an element of a tensor product of vector spaces $U \otimes V \otimes W$
- ...is a multidimensional array of numbers $A = (A_{ijk})_{i,j,k}$
- ...in $V^{\otimes d}$ is symmetric if its entries are invariant under permutations $\sigma \in \mathfrak{S}_d$
- Symmetric tensors can be identified with homogeneous polynomials (in char. 0)

$$\mathbb{C}[v_1, \ldots, v_n]_d \ni v_1 \cdots v_d \quad \mapsto \quad \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in \mathrm{Sym}^d V \subseteq V^{\otimes d}$$

Tensor decomposition and rank

- A tensor of the form $(u_i v_j w_k)_{i,j,k} \mathrel{\hat{=}} u \otimes v \otimes w$ is simple
- Every tensor is a sum of simple tensors

$$A = \sum_{i=1}^{r} \lambda_i u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$

- The smallest such r is the tensor rank of A
- Generalizes matrix rank: $A = S \cdot \mathrm{diag}(\underbrace{1, \ldots, 1}_{\mathrm{rank}\, A}, 0, \ldots, 0) \cdot T$
- If the simple tensors are unique up to scaling, then A is called identifiable
- Symmetric case: Simple tensor $v^{\otimes d} \mathrel{\hat{=}} \ell^d$, $F = \sum_{i=1}^{r} \lambda_i \ell_i^d$, symmetric tensor rank, ...

Examples

We will identify symmetric tensors with homogeneous polynomials in $T = \mathbb{C}[X_0, \dots, X_n]$.

- Rank 1 = powers of linear forms ℓ^d = cone over Veronese variety

$$V_{d,n} := \nu_d(\mathbb{P}(T_1)) \subseteq \mathbb{P}(T_d), \qquad \nu_d([\ell]) = [\ell^d]$$

  Projective space $\mathbb{P}(V) := (V \setminus 0)/\sim$, where $v \sim w$ iff $v = \lambda w$ for some $\lambda \in \mathbb{C}^\times$

- Quadratic forms = sym. matrices: $F = x^\mathsf{T} A x$, then $\operatorname{rk} F = \operatorname{rank} A$
- Fun exercise: $\operatorname{rk}(X_1^d + \cdots + X_n^d) = n$
- $\operatorname{rk}(X_0 X_1) = 2$, as $X_0 X_1 = \frac{1}{4}(X_0 + X_1)^2 - \frac{1}{4}(X_0 - X_1)^2$
- $\operatorname{rk}(X_0 X_1^{d-1}) = d$, more generally for $\alpha_0 \leq \alpha_1 \leq \dots$

$$\operatorname{rk}(X_0^{\alpha_0} \cdots X_n^{\alpha_n}) = (\alpha_1 + 1) \cdots (\alpha_n + 1)$$

- But $d X_0 X_1^{d-1} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\varepsilon X_0 + X_1)^d - \frac{1}{\varepsilon} X_1^d$, so $\{\operatorname{rk} \leq r\}$ is *not* closed

A general form walks into the door

Theorem (Alexander–Hirschowitz)

For $r(n+1) \leq \binom{n+d}{d}$ the affine cone

$$\widehat{\sigma_r V_{d,n}} = \overline{\{\, F \in T_d \mid \operatorname{rk}(F) \leq r \,\}} \subseteq \mathbb{C}[X_0, \ldots, X_n]_d \cong \mathbb{C}^{\binom{n+d}{n}}$$

has the expected (complex) dimension $r(n+1)$ except for

$$(d, n, r) = (2, \geq 2, \geq 2), (3, 4, 7), (4, 2, 5), (4, 3, 9), (4, 4, 14).$$

In particular, a general polynomial has rank $\left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$.

Running example:

A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ has $\operatorname{rk} F = \frac{1}{3}\binom{2+10}{2} = 22$. The set of such forms of rank 18 has dimension 54 in $\mathbb{C}^{66}$

General forms of subgeneric rank are identifiable

Theorem (Ballico, Mella, Chiantini–Ottaviani–Vannieuwenhoven, . . .)

For $r(n+1) < \binom{n+d}{d}$ *a general form of rank* r *is identifiable except in the cases*

$$(d, n, r) = (2, \geq 2, \geq 2), (6, 2, 9), (4, 3, 8), (3, 5, 9).$$

- For applications tensors are often of subgeneric rank $\rightsquigarrow$ generic identifiability
- A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ of rank 18 has an essentially *unique* representation

$$F = \sum_{i=1}^{18} \lambda_i \ell_i^{10}, \qquad \ell_i \in \mathbb{C}[X_0, X_1, X_2]_1$$

- Given F, how do we find the ℓ_i algorithmically?

Apolarity and eigenvalue methods

The fundamental theorem of tensor decomposition

- Let $S = \mathbb{C}[\partial_0, \dots, \partial_n]$ then S acts on $T = \mathbb{C}[X_0, \dots, X_n]$ by differentiation
$$\partial^\alpha \bullet x^\beta = \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} \text{ if } \beta \geq \alpha, \text{ else } 0$$
- S is a ring of functions on $\mathbb{P}(T_1)$ via $g([\ell]) = g \bullet \ell^{\deg g}$
- For $Z \subseteq \mathbb{P}(T_1)$ set $I(Z) = \bigoplus_{d \geq 0} \{ g \in S_d \mid g([\ell]) = 0 \text{ for } [\ell] \in Z \}$
- For $F \in T$ let $F^\perp = \operatorname{Ann}_S(F) = \{ g \in S \mid g \bullet F = 0 \}$

Theorem (Apolarity lemma)

For $F \in T_D$ and $\ell_1, \dots, \ell_r \in T_1$ the following are equivalent:

1. $F = \lambda_1 \ell_1^D + \cdots + \lambda_r \ell_r^D$ *for some* $\lambda_i \in \mathbb{C}$;
2. $I(\{[\ell_1], \dots, [\ell_r]\}) \subseteq F^\perp$ *in* S.

The Catalecticant method

- If $F = \sum_{i=1}^{r} \lambda_r \ell_1^D + \cdots + \lambda_r \ell_r^D$, then $F^\perp$ *contains* polynomials vanishing on $[\ell_i]$
- For $d \leq \frac{D}{2}$, $r < \binom{d+n}{n} - n$ and $F \in T_D$ general of rank r, then actually

$$(F^\perp)_d = I([\ell_1], \ldots, [\ell_r])_d$$

- By definition $(F^\perp)_d = \operatorname{Ker} \operatorname{Cat}_{d,D-d}(F)$ where

$$\operatorname{Cat}_{d,D-d}(F) \colon S_d \to T_{D-d}, \qquad g \mapsto g \bullet F$$

- Algorithmic approach:
  - Compute basis $\mathcal{F}$ of kernel
  - Solve polynomial system $\{\mathcal{F} = 0\}$ to get $\mathcal{Z}eros(\mathcal{F}) \overset{?}{=} \{[\ell_1], \ldots, [\ell_r]\} =: Z$,
  - Solve *linear* equations to get λ_i

⇝ When is $\mathcal{Z}eros(F_d^\perp) = Z$? Equivalently $\mathcal{Z}eros(I(Z)_d) = Z$?

Methods for polynomial system solving

Task: Given 0-dim'l system $J \subseteq S$, compute $Z = \{z_1, \dots, z_r\} = \mathcal{Z}eros(J) \subseteq \mathbb{P}^n$

- ▷ Our situation: $J = \langle I(Z)_d \rangle_S$, a chopped ideal of r general points
- ▷ (At least) three common approaches:
  - Gröbner bases computation (symbolic)
  - Homotopy continuation (numerical)
  - Eigenvalue/normal form methods (numerical/mixed)
- ▷ Gröbner bases become quickly infeasible for higher number of variables or degree
- ▷ Homotopy continuation struggles with heavily over-determined systems

⇝ Focus on the eigenvalue method approach here

Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $J \subseteq S$, compute $Z = \{z_1, \dots, z_r\} = \mathcal{Z}eros(J) \subseteq \mathbb{P}^n$

- For t large enough, $h_{S/J}(t) := \dim_{\mathbb{C}}(S/J)_t = r$ and $J_t = I(Z)_t$
- Multiplication map for $g \in S_e$:

$$M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/J)_{d+e}$$

- Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has left eigenpairs

$$\{\, (\mathrm{ev}_{z_i}, \tfrac{g}{h}(z_i)) \mid i = 1, \dots, r \,\}, \qquad \mathrm{ev}_{z_i}(f) = f(z_i)/h(z_i)$$

⇝ Translate problem into large eigenvalue problem, solve numerically

- For this need $h_{S/J}(d+e) = h_{S/J}(d) = r$, want $d, d+e$ as small as possible

Example: J saturated

If $J = I(Z)$ and Z is a general set of points, then $h_{S/I(Z)} = \min\{h_S(t), r\}$.
Hence $d = \min\{\, t \mid h_S(t) \geq r \,\}$ and $e = 1$ work.

Recap

We are lead to the following setup:

- Given a general form $F = \sum_{i=1}^{r} \lambda_i \ell_i^D \in \mathbb{C}[X_1, \ldots, X_n]_D$ of rank $r < \binom{n+\lfloor D/2 \rfloor}{n} - n$
- Decomposition is unique, want to find $Z = \{[\ell_1], \ldots, [\ell_r]\} \in \mathbb{P}^n$
- Have access to $\mathcal{F} = I(Z)_d$ only for $d \leq \frac{D}{2}$
- Want to solve polynomial system $\mathcal{F}$ using the eigenvalue method
- Is $\mathcal{Z}eros(\mathcal{F}) = Z$? With(out) multiplicities?
- What is the Hilbert function of the subideal $\langle \mathcal{F} \rangle_S \subseteq I(Z)$? When $= r$?

Running example

$n = 2,\ D = 10,\ r = 18.\ F = \sum_{i=1}^{18} \lambda_i \ell_i^{10} \in \mathbb{C}[X_0, X_1, X_2]_{10}.$
Only interesting: $d = D/2 = 5$, since for $d \leq 4$ we have $I(Z)_d = 0$!

Some nice geometry behind this!

Mathematics > Commutative Algebra

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Hilbert Functions of Chopped Ideals

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A chopped ideal is obtained from a homogeneous ideal by considering only the generators of a fixed degree. We investigate cases in which the chopped ideal defines the same finite set of points as the original one-dimensional ideal. The complexity of computing these points from the chopped ideal is governed by the Hilbert function and regularity. We conjecture values for these invariants and prove them in many cases. We show that our conjecture is of practical relevance for symmetric tensor

Rediscovering a notion introduced by [Ahmed–Fröberg–Rafiq]

Definition (Chopped ideal)

The *chopped ideal* of a homogeneous ideal $I \subseteq S$ in degree d is

$$I_{\langle d\rangle} := \langle I_d\rangle_S = \bigoplus_{t\geq d}\langle S_{t-d}\cdot I_d\rangle_{\mathbb{C}} \subseteq I \subseteq S.$$

From now on $Z \subseteq \mathbb{P}^n$ is a general set of r points, $I = I(Z)$, $d = \min\{\, t \mid \binom{n+t}{n} \geq r\,\}$.

- Min. generators of I live in degrees $\{d, d+1\}$
- Can we recover Z from $I(Z)_{\langle d\rangle}$?
- When does $(I(Z)_{\langle d\rangle})_{d+e} = I(Z)_{d+e}$?
- What is the Hilbert function $h_{I(Z)_{\langle d\rangle}}(t)$?

Example: Z = 18 points in the plane

t	...	3	4	5	6	7
$h_S(t)$	...	10	15	21	28	36
$h_I(t)$	...	0	0	3	10	18
$h_{I_{\langle 5 \rangle}}(t)$	...	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$h_S(t)$	1	3	6	10	15	21	28	36
$h_{S/I}(t)$	1	3	6	10	15	18	18	18
$h_{S/I_{\langle 5 \rangle}}(t)$	1	3	6	10	15	18	19	18

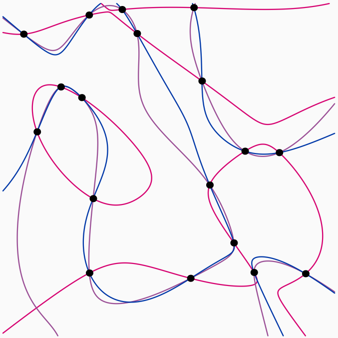
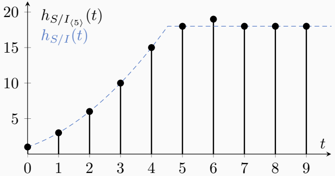
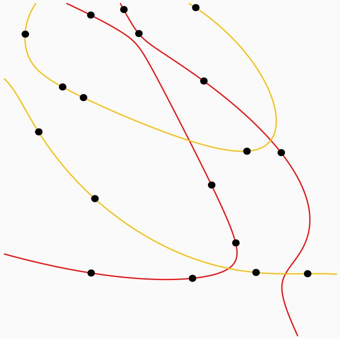


Figure 1: Three quintics $\langle q_1, q_2, q_3 \rangle_{\mathbb{C}} = I_5$ passing through 18 general points (left) and the missing split sextic $cc' \in I_6$ (right).





Recovering the points from their chopped ideal

- Generally $I_{\langle d \rangle} \subsetneq I$, but maybe

$$I \overset{?}{=} (I_{\langle d \rangle})^{\text{sat}} := \bigcup_{k \geq 0} (I_{\langle d \rangle} : \mathfrak{m}^k) \qquad \Longleftrightarrow \qquad \mathcal{Z}\textit{eros}(I) \underset{\text{multiplicities}}{\overset{?}{=}} \mathcal{Z}\textit{eros}(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$$

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$.

1. *If $r > \binom{n+d}{n} - n$, then $\mathcal{Z}eros(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
2. *If $r = \binom{n+d}{n} - n$, then $\mathcal{Z}eros(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
3. *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z without multiplicity ("reduced")*

In particular, $I = (I_{\langle d \rangle})^{\text{sat}}$ if and only if $r < \binom{n+d}{n} - n$ or $r = 1$ or $(n, r) = (2, 4)$.

Towards the expected Hilbert function – naively

- Graded components of $I_{\langle d \rangle}$ are images of multiplication map

$$\mu_e \colon S_e \otimes_{\mathbb{C}} I_d \to I_{d+e}, \qquad g \otimes f \mapsto g \cdot f$$

- One may expect μ_e to have *maximal rank*, i.e. to be injective or surjective:

$$h_{I_{\langle d \rangle}}(t) \overset{?}{=} \min\{h_I(t),\, h_S(t-d) \cdot h_I(d)\}$$

$\rightsquigarrow$ $e = 1$: Ideal generation conjecture (IGC) predicting number of minimal generators of I

- This turns out to be too optimistic; μ_e has elements in its kernel, for example

$$f_1 \otimes f_2 - f_2 \otimes f_1 \in \operatorname{Ker} \mu_d, \qquad f_1, f_2 \in I_d$$

- This *does* happen, e.g. $r = 52$ points in $\mathbb{P}^3$, then μ_5 does not have maximal rank

Towards the expected Hilbert function – more carefully

- The kernel of μ_e contains the Koszul syzygies Ksz_e generated by

$$gf_i \otimes f_j - gf_j \otimes f_i, \qquad g \in S_{e-d},\ f_i, f_j \in I_d$$

- Expecting $\operatorname{Ker}\mu_e = \mathrm{Ksz}_e$, a first estimate of $\dim_{\mathbb{C}} \operatorname{Ker}\mu_e$ is $h_S(e-d) \cdot \binom{h_I(d)}{2}$
- Expect the syzygies to also only have Koszul syzygies, correct by $h_S(e-2d) \cdot \binom{h_I(d)}{3}$
- And these also only have Koszul syzygies and ...
- This leads to the following estimate for $h_{S/I_{\langle d \rangle}}(t)$:

$$h_S(t) - \underbrace{h_S(t-d)h_I(d)}_{\text{gen's of } I_d} + \underbrace{h_S(t-2d)\binom{h_I(d)}{2}}_{\text{Koszul syzygies}} - \underbrace{h_S(t-3d)\binom{h_I(d)}{3}}_{\text{Koszul syzygy syzygies}} \pm \dots$$

- On the other hand, as soon as $h_{I_{\langle d \rangle}}(t_0) \geq h_I(t_0)$, then $I_t = (I_{\langle d \rangle})_t$ for $t \geq t_0$

The main conjecture

Expected syzygy conjecture (ESC)

$$h_{S/I_{\langle d\rangle}}(t) = \begin{cases} \sum_{k\geq 0}(-1)^k \cdot h_S(t-kd)\cdot \binom{h_I(d)}{k} & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the first integer $> d$ such that the sum is $\leq r$.

- This is always a (lexicographic) lower bound due to Fröberg
- If $W \subseteq S_d$ is a random vector subspace of dim. $h_I(d)$, then the sum is the expected Hilbert function of $S/\langle W\rangle_S$ (until sum ≤ 0)
- Proven by Nenashev in many cases, approach generalized by Blomenhofer & Casarotti

Slogan: *Chopped ideals of general points are (Fröberg-)general as long as possible*

Is the complicated alternating sum really needed?

- For $\mathbb{P}^2$ the (ESC) "actually" says $h_{I_{\langle d \rangle}}(t) = \min\{h_I(d) \cdot h_S(t-d),\, h_I(t)\}$
- This is no longer true in higher dimension – in general n summands are required
- Smallest example: 52 points in $\mathbb{P}^3$

$$h_{S/I_{\langle 5 \rangle}}(t) =$$

$$\begin{cases} h_S(t) - 4h_S(t-5) + 6h_S(t-10) & t < 11, \\ 52 & t \geq 11 \end{cases}$$

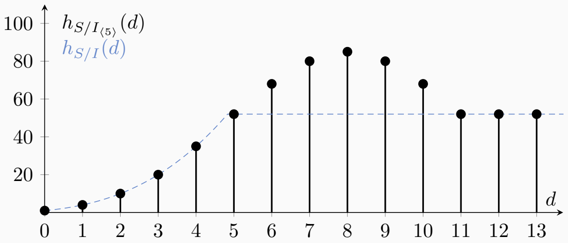


Figure 2: The Hilbert function of the chopped ideal of 52 general points in $\mathbb{P}^3$.

Main results

Theorem

Conjecture (ESC) is true in the following cases:

- $r_{\max} := h_S(d) - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ *and [*$n \leq 4$ *or generally whenever (IGC) holds].*
- *In a large number of individual cases in low dimension (next slide).*

The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

Whenever $I_{\langle d\rangle}$ *is non-saturated, one has* $\operatorname{reg}_{\mathrm{CM}} S/I_{\langle d\rangle} = \operatorname{reg}_{\mathrm{H}} S/I_{\langle d\rangle} - 1 = d + e - 1.$

Verification using computer algebra

- Testing the conjecture for particular values of (n, r):
  - Sample r random points from $\mathbb{P}^n(\mathbb{Q})$
  - Calculate $h_{S/I(Z)_{\langle d \rangle}}$ using a computer algebra system
  - If the sample satisfies (ESC), then the conjecture is true for general such Z

Theorem

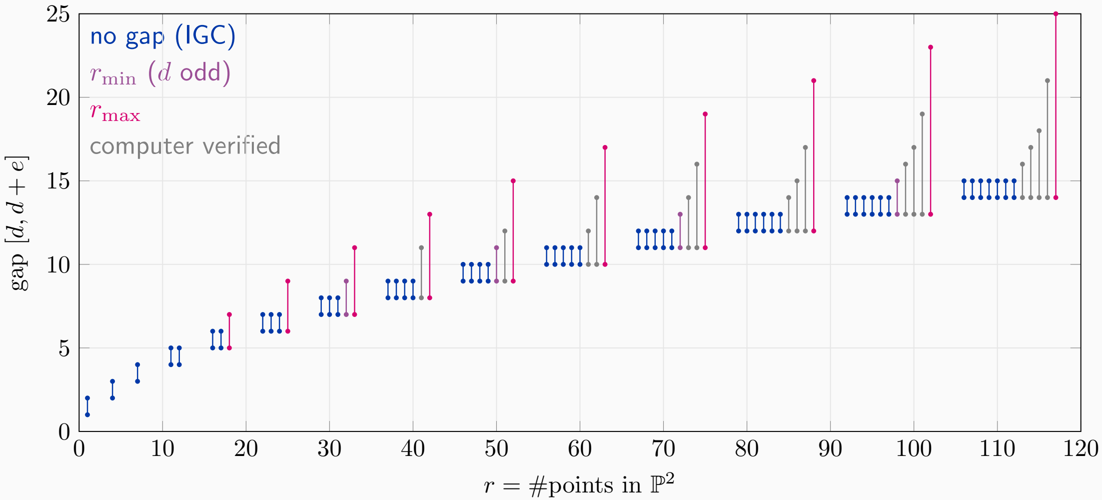
The map $Z \mapsto h_{S/I(Z)_{\langle d \rangle}}(t)$ is upper semicontinuous on the set $U \subseteq (\mathbb{P}^n)^r$ of points with generic Hilbert function.

- To speed up computation, perform calculations over a finite field $\mathbb{F}_p$
- Using `Macaulay2` we verified the conjecture in the following cases

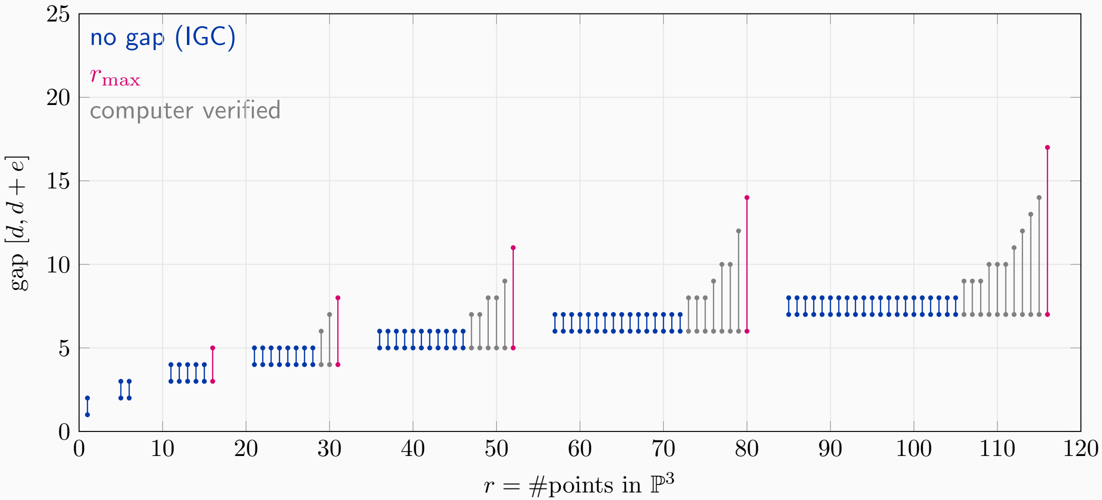
n	2	3	4	5	6	7	8	9	10
r	≤ 1825	≤ 1534	≤ 991	≤ 600	≤ 447	≤ 316	≤ 333	≤ 204	≤ 259
d	≤ 58	≤ 18	≤ 9	≤ 6	≤ 4	≤ 3	≤ 3	≤ 2	≤ 2

Visualization of the saturation gaps in $\mathbb{P}^2$

- ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is



Visualization of the saturation gaps in $\mathbb{P}^3$



Thank you! Questions?

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References i

Chwas Ahmed, Ralf Fröberg, and Mohammed Rafiq Namiq.
The graded betti numbers of truncation of ideals in polynomial rings, 2022.

A. Bernardi, E. Carlini, M. V. Catalisano, A. Gimigliano, and A. Oneto.
The hitchhiker guide to: Secant varieties and tensor decomposition.
Mathematics, 6(12):314, 2018.

Ralf Fröberg.
An inequality for hilbert series of graded algebras.
Mathematica Scandinavica, 56(2), 1985.

Fulvio Gesmundo, Leonie Kayser, and Simon Telen.
Hilbert functions of chopped ideals, 2023.

References ii

Anna Lorenzini.
The minimal resolution conjecture.
Journal of Algebra, 156(1), 1993.

Simon Telen.
Solving Systems of Polynomial Equations.
PhD thesis, KU Leuven, Leuven, Belgium, 2020.

Simon Telen and Nick Vannieuwenhoven.
A normal form algorithm for tensor rank decomposition.
ACM Trans. on Math. Soft., 48(4):1–35, 2022.

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