Hilbert functions of chopped ideals

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Motivation: Symmetric tensor decomposition

Task: Given $F \in T = \mathbb{C}[X_0, \ldots, X_n]$ of degree D, calculate decomposition

$$F = L_1^D + \cdots + L_r^D, \qquad L_i \in T_1,\ r = \mathrm{rk}(F) \text{ minimal}$$

- If $r < \binom{\lfloor D/2 \rfloor + n}{n} - n$, then generically unique

$$[F] \dashrightarrow Z = \{[L_1], \ldots, [L_r]\} \subseteq \mathbb{P}(T_1)$$

- $S := \mathbb{C}[x_0, \ldots, x_n]$ acts on T via differentiation (*Catalecticant map*)

$$C_F(d, D-d)\colon S_d \to T_{D-d}, \qquad g \mapsto g(\partial_0, \ldots, \partial_n)F$$

- $I(Z)_d \subseteq \operatorname{Ker} C_F(d, D-d)$ with equality for $d \leq \lfloor D/2 \rfloor$ and F general

$\rightsquigarrow$ Obtain low degree equations – do these suffice to recover Z?

Key example: $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ of rank 18, obtain equations of degree ≤ 5

$$\dim_{\mathbb{C}} I_5 = \dim_{\mathbb{C}} S_5 - 18 = 3$$

Motivation: Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $J \subseteq S$, compute $Z = \{z_1, \ldots, z_r\} = V(J) \subseteq \mathbb{P}^n$

- ▷ Our situation: $J = \langle I(Z)_d \rangle$, a chopped ideal of r general points
- ▷ For d large enough, $h_{S/J}(d) := \dim_{\mathbb{C}}(S/J)_d = r$ and $J_d = I(Z)_d$
- ▷ Multiplication map: $g \in S_e$, $M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/I)_{d+e}$
- ▷ Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has left eigenpairs

$$\{ (\mathrm{ev}_{z_i}, \tfrac{g}{h}(z_i)) \mid i = 1, \ldots, r \}, \qquad \mathrm{ev}_{z_i}(f) = f(z_i)/h(z_i)$$

⇝ Translate problem into large eigenvalue problem, solve numerically

- ▷ For this need $h_{S/J}(d+e) = h_{S/J}(d) = r$, want $d, d+e$ as small as possible

Goal

Study Hilbert functions of chopped ideals of general sets of points!

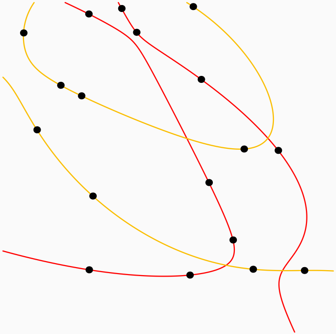
Key example: 18 points in the plane

$Z = \{z_1, \dots, z_{18}\} \subseteq \mathbb{P}^2$ general, $I := I(Z) \subseteq S$, $\langle I_5 \rangle \subsetneq I$

d	$\dots$	3	4	5	6	7
$h_I(d)$	$\dots$	0	0	3	10	18
$h_{\langle I_5 \rangle}(d)$	$\dots$	0	0	3	9	18

d	0	1	2	3	4	5	6	7
$h_{S/I}(d)$	1	3	6	10	15	18	18	18
$h_{S/\langle I_5 \rangle}(d)$	1	3	6	10	15	18	19	18

Figure 1: Three quintics $\langle q_1, q_2, q_3 \rangle_{\mathbb{C}} = I_5$ passing through 18 general points (left) and the missing split sextic $cc' \in I_6$ (right).



Chopped ideals and their saturation

- For a homogeneous ideal $I \subseteq S$ let $I_{\langle d \rangle} := \langle I_d \rangle$ be its chopped ideal in degree d
- Usually consider chop degree $d = \min \{ d \mid I_d \neq 0 \}$
- Generally $I_{\langle d \rangle} \subsetneq I$, but maybe

$$I \overset{?}{=} (I_{\langle d \rangle})^{\text{sat}} := \bigcup_{k \geq 0} (I_{\langle d \rangle} : \mathfrak{m}^k) \qquad \Longleftrightarrow \qquad \mathrm{V}(I) \underset{\text{schemes}}{\overset{?}{=}} \mathrm{V}(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$$

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and let d be the chop degree of $I = I(Z)$.

1. *If $r > \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
2. *If $r = \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
3. *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z scheme-theoretically.*

In particular, $I = (I_{\langle d \rangle})^{\text{sat}}$ if and only if $r < \binom{n+d}{n} - n$ or $r = 1$ or $(n, r) = (2, 4)$.

The expected Hilbert function of chopped ideals

From now on: Z general set of $\binom{d-1+n}{n} < r < \binom{d+n}{n} - n$ points, $I = I(Z)$

- For general points, $h_{S/I(Z)}(d) = \min\{h_S(d), r\}$, where $h_S(d) = \binom{d+n}{n}$ for $d \geq 0$
- Graded components of $I_{\langle d \rangle}$ are images of multiplication map

$$\mu_e \colon S_e \otimes_{\mathbb{C}} I_d \to I_{d+e}, \qquad g \otimes f \mapsto g \cdot f$$

- Kernel contains Koszul syzygies $f_i \otimes f_j - f_j \otimes f_i$
- Expectation: These are the only syzygies, and they have only Koszul syzygies, and ...

Expected syzygy conjecture (ESC)

For $j \geq d$ one has

$$h_{S/I_{\langle d \rangle}}(j) = \max\left\{ \sum_{k \geq 0} (-1)^k \cdot h_S(j - kd) \cdot \binom{h_S(d) - r}{k}, r \right\}.$$

This is always a lower bound due to Fröberg.

Unraveling the conjecture

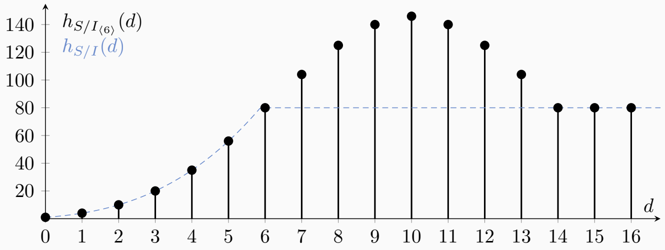
- $j = d + 1$: Predict the number of minimal generators of I in degree I_{d+1}

⇝ Ideal generation conjecture (IGC): μ_1 is injective or surjective

- For $\mathbb{P}^2$ the (ESC) says $h_{I_{\langle d \rangle}}(j) = \min\{h_I(d) \cdot h_S(j-d),\, h_I(j)\}$, i.e.

"μ_e is injective until it is surjective – always maximal rank"

- This is no longer true in higher dimension – more terms are required.
- Example: 80 points in $\mathbb{P}^3$: $h_{S/I_{\langle 6 \rangle}}(j) = \max\{h_S(j) - 4h_S(j-6) + 6h_S(j-12), 80\}$



Main results

Theorem

Conjecture (ESC) is true in the following cases:

- $r_{\max} := h_S(d) - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ *and [*$n \leq 4$ *or generally whenever (IGC) holds].*
- *In a large number of individual cases in low dimension (next slide).*

The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

Whenever $I_{\langle d\rangle}$ *is non-saturated, one has* $\operatorname{reg}_{\mathrm{CM}} S/I_{\langle d\rangle} = \operatorname{reg}_{\mathrm{H}} S/I_{\langle d\rangle} - 1 = d + e - 1$.

Verification using computer algebra

- Testing the conjecture for particular values of (n, r):
  - Sample r random points from $\mathbb{P}^n(\mathbb{Q})$
  - Calculate $h_{S/I(Z)_{\langle d \rangle}}$ using a computer algebra system
  - If the sample satisfies (ESC), then the conjecture is true for general such Z

Theorem

The map $Z \mapsto h_{S/I(Z)_{\langle d \rangle}}(j)$ is upper semicontinuous on the set $U \subseteq (\mathbb{P}^n)^r$ of points with generic Hilbert function.

- To speed up computation, perform calculations over a finite field $\mathbb{F}_p$
- Using `Macaulay2` we verified the conjecture in the following cases

n	r	d
2	≤ 1203	≤ 47
3	≤ 642	≤ 13
4	≤ 289	≤ 6

Visualization of the saturation gaps

- ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is

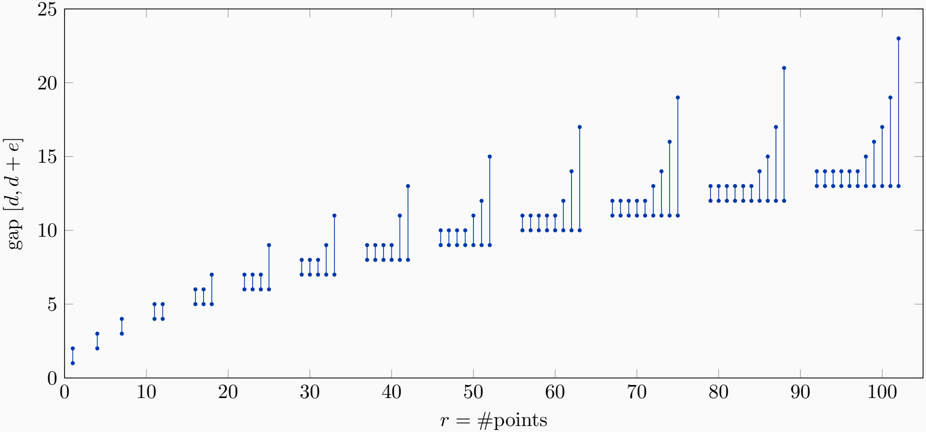


Figure 3: The saturation gaps for all values of $r \leq 102$ in $\mathbb{P}^2$.

Outlook

- Characteristic $p > 0$? $\rightsquigarrow$ Yes (mostly).
- Proving the conjecture in $\mathbb{P}^2$?
- Improve code to verify more cases
- Generalizations multi-graded setting, e.g. points in $\mathbb{P}^n \times \mathbb{P}^m$
- State a conjecture for the minimal free resolution of $I(Z)_{\langle d \rangle}$

Thank you! Questions?

Preprint soon™

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