Hilbert functions of chopped ideals

Nonlinear Algebra Seminar

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

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Motivation: Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $J \subseteq S = \mathbb{C}[x_0, \dots, x_n]$, compute $Z = \{z_1, \dots, z_r\} = \mathrm{V}(J) \subseteq \mathbb{P}^n$

- For t large enough, $h_{S/J}(t) := \dim_{\mathbb{C}}(S/J)_t = r$ and $J_t = I(Z)_t$
- Multiplication map: $g \in S_e$, $M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/J)_{d+e}$
- Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has left eigenpairs

$$\left\{ (\mathrm{ev}_{z_i}, \tfrac{g}{h}(z_i)) \mid i = 1, \dots, r \right\}, \qquad \mathrm{ev}_{z_i}(f) = f(z_i)/h(z_i)$$

$\rightsquigarrow$ Translate problem into large eigenvalue problem, solve numerically

- For this need $h_{S/J}(d+e) = h_{S/J}(d) = r$, want $d, d+e$ as small as possible

Example: J saturated

If $J = I(Z)$ and Z is a general set of points, then $h_{S/I(Z)} = \min\{h_S(t), r\}$.

Hence $d = \min\{ t \mid h_S(t) \geq r \}$ and $e = 1$ work.

Motivation: Symmetric tensor decomposition

Task: Given $F \in T = \mathbb{C}[X_0, \dots, X_n]$ of degree D, calculate decomposition

$$F = L_1^D + \cdots + L_r^D, \qquad L_i \in T_1,\ r = \mathrm{rk}(F) \text{ minimal}$$

- ▷ If $r < h_S(\lfloor \frac{D}{2} \rfloor) - n$, then generically unique summands

$$[F] \dashrightarrow Z = \{[L_1], \dots, [L_r]\} \subseteq \mathbb{P}(T_1)$$

- ▷ Equations of Z are contained in the kernel of the *Catalecticant map*

$$C_F(d, D-d)\colon S_d \to T_{D-d}, \qquad g \mapsto g(\partial_0, \dots, \partial_n)F$$

- ▷ $I(Z)_d \subseteq \operatorname{Ker} C_F(d, D-d)$ with equality for $d \leq \lfloor \frac{D}{2} \rfloor$ and F general

$\rightsquigarrow$ Obtain all equations on Z in a single *low* degree d

Key example: $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ of rank 18, obtain equations of degree ≤ 5

The funny word in the title

Definition (Chopped ideal)

The *chopped ideal* of a homogeneous ideal $I \subseteq S$ in degree d is $I_{\langle d \rangle} := \langle I_d \rangle_S$.

From now on $Z \subseteq \mathbb{P}^n$ is a general set of r points, $I = I(Z)$, $d = \min\{\, t \mid h_S(t) \geq r \,\}$.

- Can we recover Z from $I(Z)_{\langle d \rangle}$?
- When does $(I(Z)_{\langle d \rangle})_{d+e} = I(Z)_{d+e}$?
- What is the Hilbert function $h_{I(Z)_{\langle d \rangle}}(t)$?

Example: Z = 18 points in the plane

t	$\dots$	3	4	5	6	7
$h_S(t)$	$\dots$	10	15	21	28	36
$h_I(t)$	$\dots$	0	0	3	10	18
$h_{I_{\langle 5\rangle}}(t)$	$\dots$	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$h_S(t)$	1	3	6	10	15	21	28	36
$h_{S/I}(t)$	1	3	6	10	15	18	18	18
$h_{S/I_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	19	18

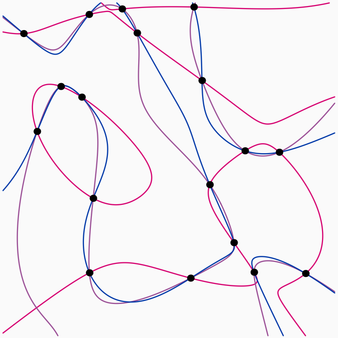
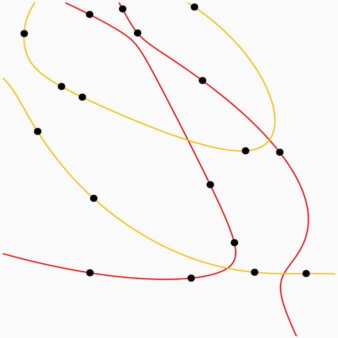


Figure 1: Three quintics $\langle q_1, q_2, q_3\rangle_{\mathbb{C}} = I_5$ passing through 18 general points (left) and the missing split sextic $cc' \in I_6$ (right).



Recovering the points from their chopped ideal

- Generally $I_{\langle d \rangle} \subsetneq I$, but maybe

$$I \overset{?}{=} (I_{\langle d \rangle})^{\mathrm{sat}} := \bigcup_{k \geq 0} (I_{\langle d \rangle} : \mathfrak{m}^k) \qquad \Longleftrightarrow \qquad \mathrm{V}(I) \underset{\text{schemes}}{\overset{?}{=}} \mathrm{V}(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$$

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$.

1. *If $r > \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
2. *If $r = \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
3. *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z scheme-theoretically.*

In particular, $I = (I_{\langle d \rangle})^{\mathrm{sat}}$ if and only if $r < \binom{n+d}{n} - n$ or $r = 1$ or $(n, r) = (2, 4)$.

Towards the expected Hilbert function

- Graded components of $I_{\langle d \rangle}$ are images of multiplication map

$$\mu_e \colon S_e \otimes_{\mathbb{C}} I_d \to I_{d+e}, \qquad g \otimes f \mapsto g \cdot f$$

- One may expect μ_e to have *maximal rank*, i.e. to be injective or surjective:

$$h_{I_{\langle d \rangle}}(t) \overset{?}{=} \min\{h_I(t),\, h_S(t-d) \cdot h_I(d)\}$$

$\rightsquigarrow$ $e = 1$: Ideal generation conjecture (IGC) predicting number of minimal generators of I

- This turns out to be too optimistic; μ_e has elements in its kernel, for example

$$f_1 \otimes f_2 - f_2 \otimes f_1 \in \operatorname{Ker} \mu_d, \qquad f_1, f_2 \in I_d$$

- This *does* happen, e.g. $r = 52$ points in $\mathbb{P}^3$, then μ_5 does not have maximal rank

Thank you! Questions?

Better luck next time ;(

Towards the expected Hilbert function – for real

- The kernel of μ_e contains the Koszul syzygies Ksz_e generated by

$$gf_i \otimes f_j - gf_j \otimes f_i, \qquad g \in S_{e-d},\ f_i, f_j \in I_d$$

- Expecting $\operatorname{Ker}\mu_e = \mathrm{Ksz}_e$, a first estimate of $\dim_{\mathbb{C}} \operatorname{Ker}\mu_e$ is $h_S(e-d)\cdot\binom{h_I(d)}{2}$
- Expect the syzygies to also only have Koszul syzygies, correct by $h_S(e-2d)\cdot\binom{h_I(d)}{3}$
- And these also only have Koszul syzygies and ...
- This leads to the following estimate for $h_{S/I_{\langle d\rangle}}(t)$:

$$h_S(t) - \underbrace{h_S(t-d)h_I(d)}_{\text{gen's of } I_d} + \underbrace{h_S(t-2d)\binom{h_I(d)}{2}}_{\text{Koszul syzygies}} - \underbrace{h_S(t-3d)\binom{h_I(d)}{3}}_{\text{Koszul syzygy syzygies}} \pm \dots$$

- On the other hand, as soon as $h_{I_{\langle d\rangle}}(t_0) \geq h_I(t_0)$, then $I_t = (I_{\langle d\rangle})_t$ for $t \geq t_0$

The main conjecture

Expected syzygy conjecture (ESC)

$$h_{S/I_{\langle d\rangle}}(t) = \begin{cases} \sum_{k\geq 0} (-1)^k \cdot h_S(t-kd) \cdot \binom{h_I(d)}{k} & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the least integer $> d$ such that the sum is at most r.

- This is always a lower bound due to Fröberg
- Alternative expression for the ideal:

$$h_{I_{\langle d\rangle}}(t) = \begin{cases} \sum_{k\geq 1} (-1)^{k-1} \cdot h_S(t-kd) \cdot \binom{h_I(d)}{k} & t < t_0, \\ h_I(t) & t \geq t_0, \end{cases}$$

Is the complicated alternating sum really needed?

- For $\mathbb{P}^2$ the (ESC) "actually" says $h_{I_{\langle d\rangle}}(t) = \min\{h_I(d)\cdot h_S(t-d),\, h_I(t)\}$
- This is no longer true in higher dimension – in general n summands are required
- Smallest example: 52 points in $\mathbb{P}^3$

$$h_{S/I_{\langle 5\rangle}}(t) = \begin{cases} h_S(t) - 4h_S(t-5) + 6h_S(t-10) & t < 11, \\ 52 & t \geq 11 \end{cases}$$

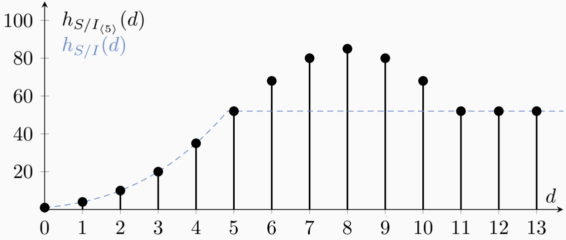


Figure 2: The Hilbert function of the chopped ideal of 52 general points in $\mathbb{P}^3$.

Theorem

Conjecture (ESC) is true in the following cases:

- $r_{\max} := h_S(d) - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ *and [*$n \leq 4$ *or generally whenever (IGC) holds].*
- *In a large number of individual cases in low dimension (next slide).*

The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

Whenever $I_{\langle d\rangle}$ *is non-saturated, one has* $\mathrm{reg}_{\mathrm{CM}}\, S/I_{\langle d\rangle} = \mathrm{reg}_{\mathrm{H}}\, S/I_{\langle d\rangle} - 1 = d + e - 1$.

Verification using computer algebra

- Testing the conjecture for particular values of (n, r):
  - Sample r random points from $\mathbb{P}^n(\mathbb{Q})$
  - Calculate $h_{S/I(Z)_{\langle d\rangle}}$ using a computer algebra system
  - If the sample satisfies (ESC), then the conjecture is true for general such Z

Theorem

The map $Z \mapsto h_{S/I(Z)_{\langle d\rangle}}(t)$ is upper semicontinuous on the set $U \subseteq (\mathbb{P}^n)^r$ of points with generic Hilbert function.

- To speed up computation, perform calculations over a finite field $\mathbb{F}_p$
- Using `Macaulay2` we verified the conjecture in the following cases

n	2	3	4	5	6	7	8	9	10
r	≤ 1825	≤ 1534	≤ 991	≤ 600	≤ 447	≤ 316	≤ 333	≤ 204	≤ 259
d	≤ 58	≤ 18	≤ 9	≤ 6	≤ 4	≤ 3	≤ 3	≤ 2	≤ 2

Visualization of the saturation gaps in $\mathbb{P}^2$

- ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is

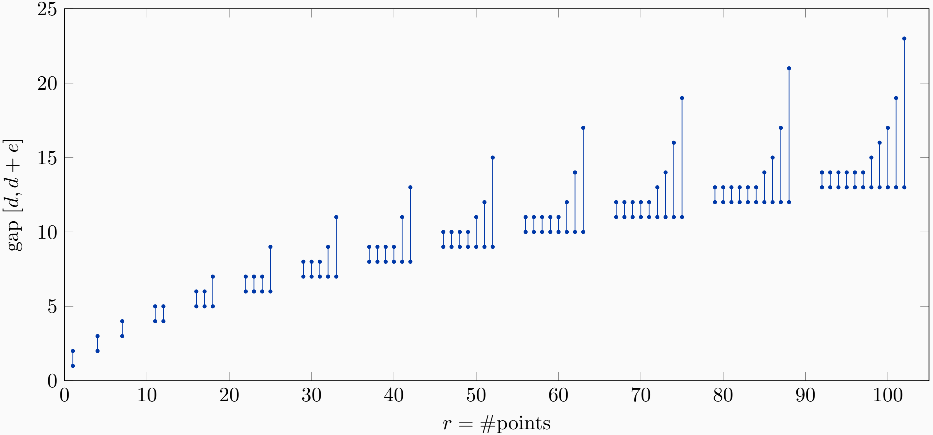


Figure 3: The saturation gaps for all values of $r \leq 102$ in $\mathbb{P}^2$.

Visualization of the saturation gaps in $\mathbb{P}^3$

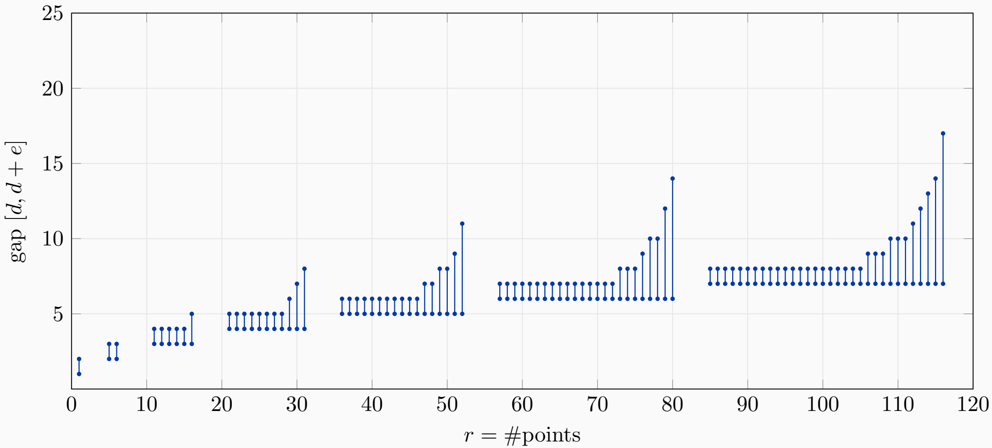


Figure 4: The saturation gaps for all values of $r \leq 116$ in $\mathbb{P}^3$.

Outlook

- Characteristic $p > 0$? $\rightsquigarrow$ Should carry over.
- Proving the conjecture in $\mathbb{P}^2$?
- Improve code to verify more cases
- Generalizations multi-graded setting, e.g. points in $\mathbb{P}^n \times \mathbb{P}^m$
- State a conjecture for the minimal free resolution of $I(Z)_{\langle d \rangle}$

Thank you! Questions?

Preprint soon™

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