Hilbert Functions of Chopped Ideals

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Motivation: Tensor decomposition using eigenvalue methods

In symmetric tensor decomposition the goal is to decompose a degree D form

$$F \in T = \mathbb{C}[X_0, \dots, X_n] \qquad \text{as} \qquad F = L_1^D + \dots + L_r^D$$

with the minimal number of powers of linear forms. Considering the low rank case $r < \binom{n+\lfloor D/2 \rfloor}{n} - n$, this decomposition is generically unique.

The points $Z = \{[L_1], \dots, [L_r]\} \subseteq \mathbb{P}(T_1)$ are solutions to an over-determined system of polynomial equations arising from the *Catalecticant matrix* of F. These equations generate a sub-ideal of the vanishing ideal $I(Z)$, generated in a single degree.

In order to numerically solve these systems, one can employ *numerical normal form methods* [3]. These transform zero-dimensional systems of polynomial equations into eigenvalue problems. The complexity of this algorithms is governed by the Hilbert regularity of the equations at hand.

We propose the values of the **Hilbert functions of** *these* ***chopped ideals***.

This procedure has been implemented in `Julia`. It computes the decomposition of a general rank $r = 400$ form of degree $D = 12$ in $n + 1 = 6$ variables with 10 digits of accuracy within 25 seconds on a MacBook Pro with an Intel Core i7 processor.

Chopped ideals

Let $S \coloneqq \mathbb{C}[x_0, \dots, x_n]$ be a polynomial ring and $0 \neq I \subseteq S$ a homogeneous ideal.

Definition. The *chopped ideal* of I in degree d is the ideal $I_{\langle d \rangle} \coloneqq \langle I_d \rangle_S$ generated by the elements in degree d. We usually consider d to be the lowest degree with $I_d \neq 0$.

Now let $I = I(Z)$ be the ideal of a finite set of points $Z = \{z_1, \dots, z_r\} \subseteq \mathbb{P}^n(\mathbb{C})$. In chopping the ideal, we consider only the equations on Z of a fixed degree. The goal is to recover Z from $I_{\langle d \rangle}$. Generally $I_{\langle d \rangle} \subsetneq I$, but one may hope that

$$I(Z) \overset{?}{=} (I_{\langle d \rangle})^{\mathrm{sat}} \coloneqq (I_{\langle d \rangle} : \mathfrak{m}^\infty), \qquad \mathfrak{m} = \langle x_0, \dots, x_n \rangle_S.$$

This is the case if and only if Z is scheme-theoretically cut out by $I(Z)_d$.

Theorem. *Let $Z \subseteq \mathbb{P}^n$ be a general collection of r points with ideal $I = I(Z)$ and $d > 0$.*

- *If $r > \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
- *If $r = \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
- *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z scheme-theoretically.* $\square$

In particular, $(I_{\langle d \rangle})^{\mathrm{sat}} = I$ if and only if $r < \binom{n+d}{n} - n$ or $r = 1$ or $(n, r) = (2, 4)$.

Example: 18 points in the plane

The smallest interesting example occurs for $r = 18$ general points Z in the plane (see also fig. 4). In the tensor setting, this corresponds to a general rank 18 form of degree 10 in 3 variables. The components of $I = I(Z)$ have dimension

$$\dim_{\mathbb{C}} I_t = \max\left\{ \binom{t+2}{2} - 18 \,,\, 0 \right\}.$$

The lowest nonzero component I_5 is generated by $\binom{5+2}{2} - 18 = 3$ quintics q_1, q_2, q_3. These polynomials do *not* generate I; indeed

$$\dim_{\mathbb{C}} I_6 = 10 > 3 \cdot \dim_{\mathbb{C}} S_1 \geq \dim_{\mathbb{C}} (I_{\langle 5 \rangle})_6.$$

It turns out that $(I_{\langle 5 \rangle})_7 = I_7$; this is possible since

$$\dim_{\mathbb{C}} I_7 = 18 = 3 \cdot \dim_{\mathbb{C}} S_2.$$

The Hilbert functions of the two ideals are displayed in fig. 1.

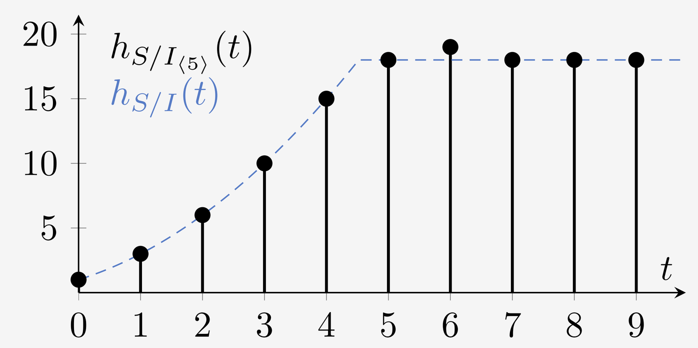

Figure 1. The Hilbert function of the chopped ideal of 18 points in $\mathbb{P}^2$.

Which generator of I_6 is missing? Splitting the points in groups of $9 + 9$, there is a unique cubic through each group. Their product spans a complement of $(I_{\langle 5 \rangle})_6$ in I_6.

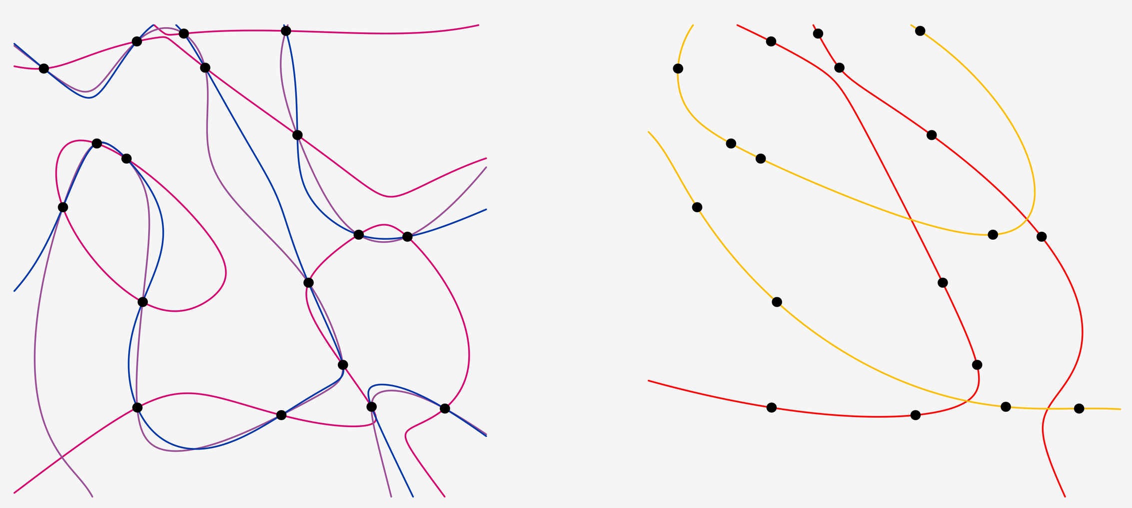

Figure 2. Three quintics (left) and a split sextic (right) through 18 points.

Hilbert functions

The *Hilbert function* of a finite graded S-module M is $h_M(t) \coloneqq \dim_{\mathbb{C}} M_t$. The Hilbert function of a set of r points $h_Z \coloneqq h_{S/I(Z)}$ is non-decreasing towards r. If Z is in general position, then

$$h_Z(t) = \min\{\, h_S(t) \,,\, r \,\}, \qquad h_S(t) = \max\left\{ \binom{n+t}{n} \,,\, 0 \right\}.$$

By the previous theorem, for general Z (with $r < h_S(d) - n$) one has

$$(I_{\langle d \rangle})_d = I_d, \qquad (I_{\langle d \rangle})_t = I_t, \quad t \gg 0,$$

since saturation alters only finitely many components.

Question. *What are the values of $h_{S/I_{\langle d \rangle}}$ in between? How large is the saturation gap?*

The components of $I_{\langle d \rangle}$ are the images under the multiplication maps

$$\mu_e \colon S_e \otimes I_d \to I_{d+e}.$$

Assuming this linear map has the largest rank possible, one expects:

Conjecture (Expected syzygy conjecture). *For Z general the Hilbert function is*

$$h_{S/I_{\langle d \rangle}}(t) = \sum_{k \geq 0} (-1)^k \cdot h_S(t - kd) \cdot \binom{h_S(d) - r}{k} \tag{1}$$

until this sum is less than or equal to r for $t_0 > d$, from which point on it stabilizes at r.

This expression is always a (lexicographic) lower bound due to a famous theorem of Fröberg [1]. The case $t = d + 1$ is equivalent to the *Ideal Generation Conjecture* (IGC) [2]; our conjecture is a natural generalization thereof.

Main results

Theorem. *Conjecture (1) is true in the following cases:*

- *$r_{\max} \coloneqq \binom{n+d}{n} - (n+1)$ for all d in all dimensions n.*
- *In the plane for $r_{\min} = \frac{1}{2}(d+1)^2$ when d is odd.*
- *$r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ and $n \leq 4$, or generally whenever (IGC) holds.*
- *In a large number of individual cases in low dimension, see next section.*

Furthermore, the length of the saturation gap is bounded above by

$$e_0 = \min\{\, e > 0 \mid (I_{\langle d \rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1). \qquad \square$$

The alternating sum in (1) is really necessary, as witnessed by fig. 3 ($e_0 > d$).

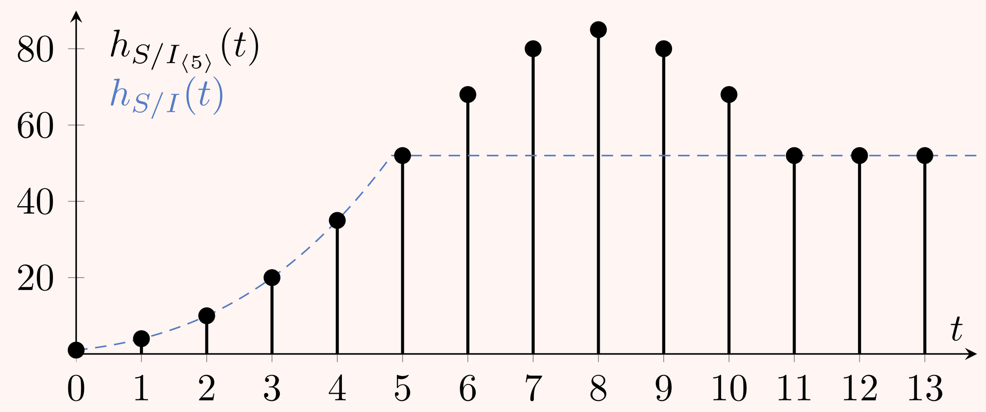

Figure 3. The Hilbert function of the chopped ideal of 52 points in $\mathbb{P}^3$.

Theorem. *Whenever $I_{\langle d \rangle}$ is non-saturated, one has $\mathrm{reg}_{\mathrm{CM}}\, S/I_{\langle d \rangle} = \mathrm{reg}_{\mathrm{H}}\, S/I_{\langle d \rangle} - 1$.* $\square$

Here $\mathrm{reg}_{\mathrm{CM}}$ is the *Castelnuovo-Mumford regularity* and $\mathrm{reg}_{\mathrm{H}}$ is the degree from which on the Hilbert function becomes constant.

The methods used in the proofs include syzygies, liaison (in higher codimension), the mapping cone construction, Hilbert-Burch theory and local cohomology.

Verification using computer algebra

To test the conjecture, one can randomly sample r points from $\mathbb{P}^n(\mathbb{Q})$ and calculate the Hilbert function using computer algebra. If the sample satisfies (1), then by the following theorem the conjecture holds true for this r.

Theorem. *For fixed t the map $Z \mapsto h_{S/I(Z)_{\langle d \rangle}}(t)$ is upper semicontinuous on the set $U \subseteq (\mathbb{P}^n)^r$ of points with generic Hilbert function.* $\square$

To speed up the computation, it suffices to verify the conjecture over a finite field $\mathbb{F}_p$. Using `Macaulay2` we verified the conjecture in the following ranges

n	2	3	4	5	6	7	8	9	10
r	≤ 2343	≤ 2296	≤ 1815	≤ 1260	≤ 904	≤ 760	≤ 479	≤ 207	≤ 267

The following figure shows the distributions of the gap lengths for various r in the plane.

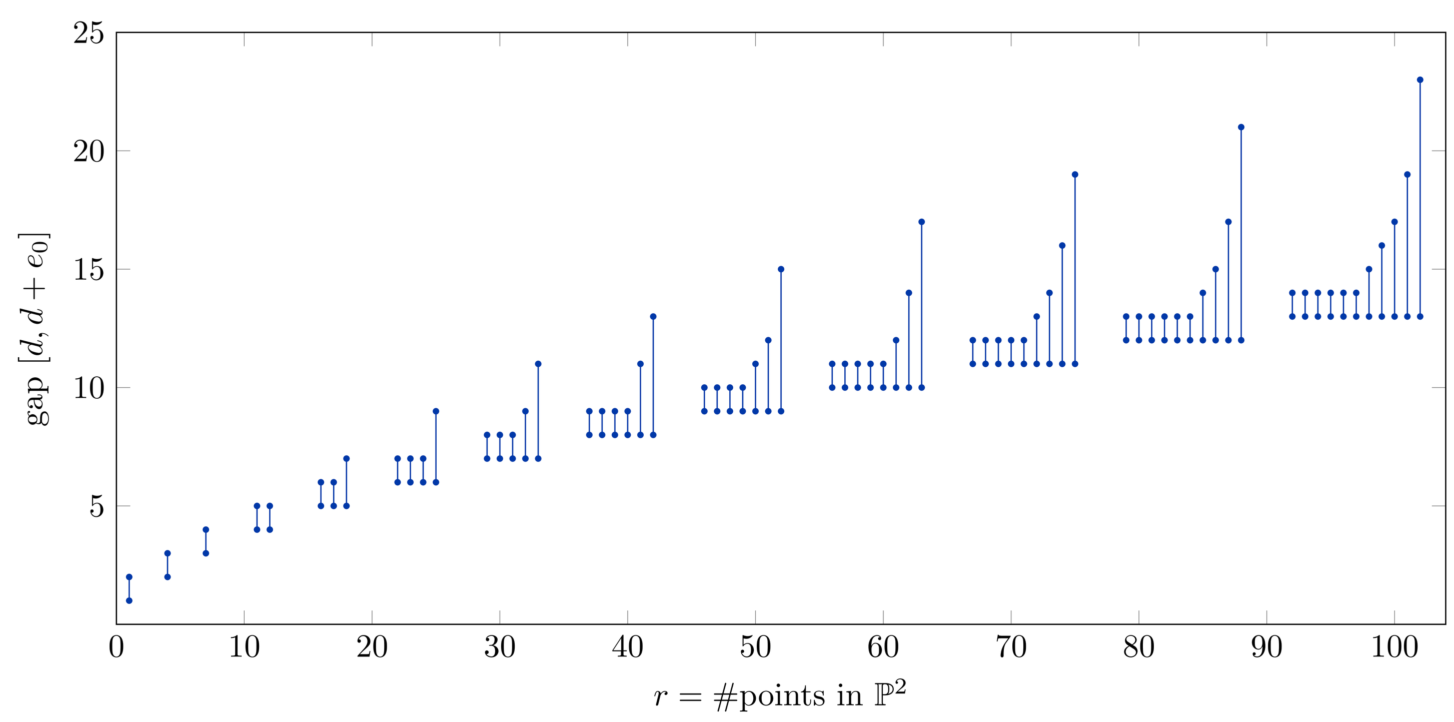

Figure 4. The saturation gaps for all values of $r \leq 102$ in the plane.

References

[1] R. Fröberg, "An inequality for hilbert series of graded algebras," *Mathematica Scandinavica*, vol. 56, no. 2, 1985.

[2] A. Lorenzini, "The minimal resolution conjecture," *Journal of Algebra*, vol. 156, no. 1, 1993.

[3] S. Telen, "Solving systems of polynomial equations," Ph.D. dissertation, KU Leuven, Leuven, Belgium, 2020.