Hilbert Functions of Chopped Ideals

STADIUS guest seminar

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

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Mathematics > Commutative Algebra

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Hilbert Functions of Chopped Ideals

Fulvio Gesmundo, Leonie Kayser, Simon Telen

A chopped ideal is obtained from a homogeneous ideal by considering
cases in which the chopped ideal defines the same finite set of points a
computing these points from the chopped ideal is governed by the Hilb
these invariants and prove them in many cases. We show that our conj
decomposition.

(Symmetric) tensor decomposition

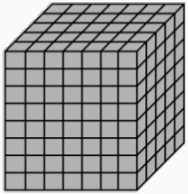
Apolarity and eigenvalue methods

A tale of 18 points in the plane

Hilbert functions of chopped ideals!

(Symmetric) tensor decomposition

What is a tensor?



A tensor...

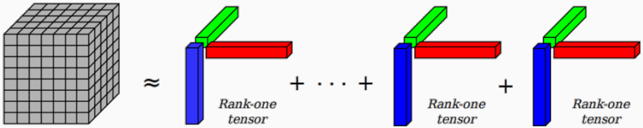
- ...is an object that transforms like a tensor
- ...is an element of a tensor product of vector spaces $U \otimes V \otimes W$
- ...is a **multidimensional array of numbers** $A = (A_{i_1 \dots i_d})_{i_1,\dots,i_d} \in \mathbb{C}^{n_1 \times \dots \times n_d}$
- ...in $(\mathbb{C}^n)^{\otimes d}$ is symmetric if its entries are invariant under permutations $\sigma \in \mathfrak{S}_d$
- Symmetric tensors can be identified with homogeneous polynomials

$$\mathbb{C}[x_1, \dots, x_n]_d \ni x_{i_1} \cdots x_{i_d} \quad \longleftrightarrow \quad \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(d)}} \in \operatorname{Sym}^d \mathbb{C}^n \subseteq (\mathbb{C}^n)^{\otimes d}$$

Tensor decomposition and rank

- A tensor of the form $(u_i v_j w_k)_{i,j,k} \hat{=} u \otimes v \otimes w$ is simple
- Every tensor is a linear combination of simple tensors

$$A = \sum_{i=1}^{r} \lambda_i u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$



- The smallest such r is the tensor rank of A
- Generalizes matrix rank: $\mathbb{C}^{m \times n} \ni A = S \cdot \mathrm{diag}(\underbrace{1, \ldots, 1}_{\text{rank } A}, 0, \ldots) \cdot T = \sum_{i=1}^{r} S_{*,i} \cdot T_{i,*}$
- If the simple tensors are unique up to scaling, then A is called identifiable
- Symmetric case: Simple tensor $v^{\otimes d} \hat{=} L^d$ powers of linear forms, $F = \sum_{i=1}^{r} \lambda_i L_i^d$
- Symmetric tensor rank, identifiability, ...

Forms of small rank often have unique decompositions

Let $T_d = \mathbb{C}[X_0, \ldots, X_n]_d \cong \mathbb{C}^{\binom{n+d}{n}}$ be the vector space of degree d forms

- **(Alexander–Hirschowitz)**
  A general form $F \in T_d$ has rank $\left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$ except in a few cases
- **(Ballico, Mella, Chiantini–Ottaviani–Vannieuwenhoven, . . .)**
  For $r < \frac{1}{n+1}\binom{n+d}{d}$ a general form of rank r is identifiable except in a few cases

Running example

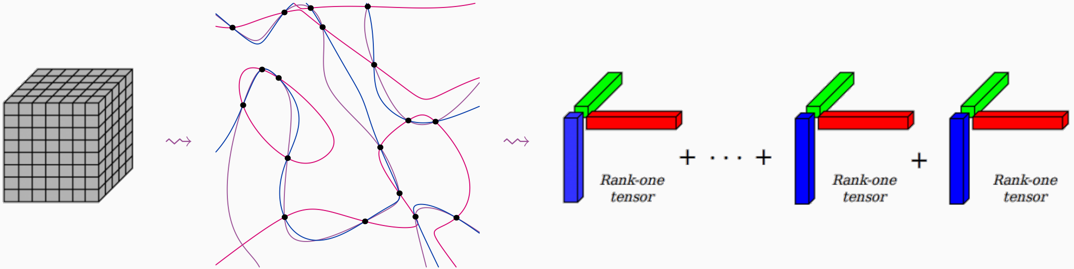
A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ has $\operatorname{rk} F = \frac{1}{3}\binom{2+10}{2} = 22$. The set of such forms of rank 18 has dimension 54 in $\mathbb{C}^{66}$. A random such F has a *unique* decomposition

$$F = L_1^{10} + \cdots + L_{18}^{10}, \qquad L_i \in \mathbb{C}[X_0, X_1, X_2]_1.$$

Apolarity and eigenvalue methods

The catalecticant method

- ▷ Assume general $F = \sum_{i=1}^{r} L_i^d \in T_d$ of rank r
- ▷ Linear forms as points in projective space $[L_i] \in \mathbb{P}(T_1) = \mathbb{P}^n_{\mathbb{C}}$ $\qquad \mathbb{P}(V) = (V \setminus 0)/\mathbb{C}^\times$
- ▷ Catalecticant method yields polynomials vanishing on $Z = \{[L_1], \ldots, [L_r]\} \subseteq \mathbb{P}^n$



- ▷ In fact: Obtain *all* homog. equations of degree $\leq d/2$ vanishing on Z

$\rightsquigarrow$ Hope: Solutions to equations are exactly the $[L_i]$!

The algorithm

- Equations via kernel of catalecticant maps $\mathrm{Cat}_j(F)$
- Algorithmic approach:
  1. Compute kernel basis $\mathcal{F}$ of the *linear* catalecticant map $\mathrm{Cat}_{\lfloor d/2 \rfloor}(F)$
  2. Solve *polynomial* system $\{\mathcal{F} = 0\}$ to get $\mathcal{Z}eros(\mathcal{F}) \stackrel{?}{=} \{[L_1], \ldots, [L_r]\}$,
  3. Solve *linear* equations to get λ_i in $F = \sum_{i=1}^{r} \lambda_i L_i^d$
- (At least) three common approaches:
  - Gröbner bases computation (symbolic)
  - Homotopy continuation (numerical)
  - Eigenvalue/normal form methods (numerical/mixed)

$\rightsquigarrow$ Focus on the eigenvalue method approach here

Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $\{\mathcal{F} = 0\}$, compute finite set $Z = \{z_1, \dots, z_r\} = \mathcal{Z}eros(\mathcal{F}) \subseteq \mathbb{P}^n$

- ▷ Consider ideal $J := \langle \mathcal{F} \rangle_S = \bigoplus_{t \geq 0} J_t$, this is a graded subspace of S with

$$J_t = S_{t - \deg f_1} f_1 + \cdots + S_{t - \deg f_s} f_s \subseteq S_t$$

- ▷ For t large enough the Hilbert function $\mathrm{hf}_{S/J}(t) := \dim_{\mathbb{C}} (S/J)_t$ is constant r
- ▷ Multiplication map for $g \in S_e$:

$$M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/J)_{d+e}$$

- ▷ Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has left eigenpairs

$$\{\, (\mathrm{ev}_{z_i}, \tfrac{g}{h}(z_i)) \mid i = 1, \dots, r \,\}, \qquad \mathrm{ev}_{z_i}(f) = f(z_i)/h(z_i)$$

$\rightsquigarrow$ Translate problem into large eigenvalue problem, solve numerically

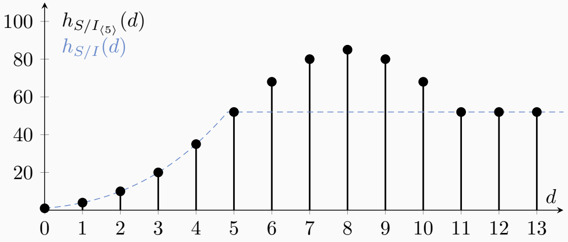
- ▷ For this need $\mathrm{hf}_{S/J}(d+e) = \mathrm{hf}_{S/J}(d) = r$, want $d, d+e$ as small as possible

Non-saturated systems are harder to solve

- Z set of points, $I = \{ f \in S \mid f(Z) = 0 \}$, then $h_{S/I}(t)$ is increasing
- If Z general, then $\mathrm{hf}_{S/I}(t) = \min\{\binom{n+t}{n}, r\} \rightsquigarrow d = \min\{ t \mid \binom{n+t}{n} \geq r \}$ and $e = 1$
- For "incomplete" $J \subsetneq I(Z)$ larger saturation gap e can be encountered
- Saturation gap governs *algorithmic complexity* of solving J with eigenvalue methods

Bigger example

For a general set $Z \subseteq \mathbb{P}^3$ of 52 points and $J = I(Z)_{\langle 5 \rangle} := \langle \{ f \in S_5 \mid f(Z) = 0 \} \rangle_S$, we have the Hilbert function pictured below. Smallest choice: $d = 5$, $d + e = 11$.



A tale of 18 points in the plane

How many forms vanish on general points?

Running example

$n = 2$, $d = 10$, $r = 18$, equations $\mathcal{F}$ have degree $d/2 = 5$.

$$F = \sum_{i=1}^{18} L_i^{10} \in \mathbb{C}[X_0, X_1, X_2]_{10}, \qquad [L_i] \in \mathbb{P}(\mathbb{C}[X_0, X_1, X_2]_1) = \mathbb{P}^2$$

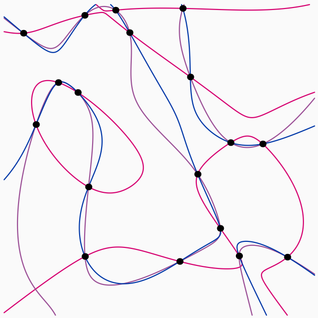
Decomposition is unique, want to find $Z = \{[L_1], \dots, [L_{18}]\} \in \mathbb{P}^2$

- For r general points $Z \subseteq \mathbb{P}^n$, $\dim I(Z)_t = \max\{0, \binom{n+t}{n} - r\}$
- Here: $21 - 18 = 3$ Equations $\mathcal{F} = \{q_1, q_2, q_3\}$, of lowest degree 5
- $28 - 18 = 10$ equations of degree 6, $26 - 18 = 18$ equations of degree 7,...
- $\dim_{\mathbb{C}} \langle S_1 \cdot \mathcal{F} \rangle_{\mathbb{C}} \leq 3 \cdot 3 = 9$, $\dim_{\mathbb{C}} \langle S_2 \cdot \mathcal{F} \rangle_{\mathbb{C}} \leq 3 \cdot 6 = 18$

Example: $Z = 18$ points in the plane

t	…	3	4	5	6	7
$\mathrm{hf}_S(t)$	…	10	15	21	28	36
$\mathrm{hf}_I(t)$	…	0	0	3	10	18
$\mathrm{hf}_{I_{\langle 5\rangle}}(t)$	…	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$\mathrm{hf}_S(t)$	1	3	6	10	15	21	28	36
$\mathrm{hf}_{S/I}(t)$	1	3	6	10	15	18	18	18
$\mathrm{hf}_{S/I_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	19	18



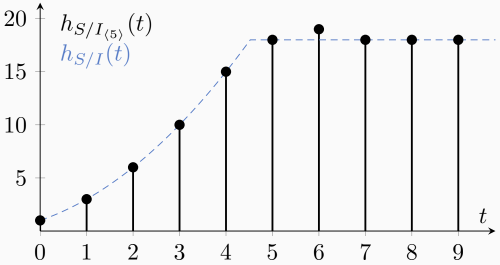


Figure 1: Three quintics $\langle q_1, q_2, q_3\rangle_{\mathbb{C}} = I_5$ passing through 18 general points.

Macaulay2 meets

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                     version 1.0.0

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                      PROBLEM INFO
  system of 3 multivariate polynomials with maximum total degree 5

    p1(x1,...,x2) = 0     <- d1 = 5
    p2(x1,...,x2) = 0     <- d2 = 5
    p3(x1,...,x2) = 0     <- d3 = 5

 sparsity factor is 0.90
------------------------------------------------------------------------

                    ALGORITHM OUTPUT
 using a Macaulay matrix algorithm with a sparse construction of
 the null space and block-wise shifts

 | degree         | nullity         | increase        | gap zone       |
 |--------------------------------------------------------------------|
 | 5              | 18              | 18              | NaN            |
 | 6              | 19              | 1               | NaN            |
 | 7              | 18              | -1              | NaN            |
 | 8              | 18              | 0               | 3              |
------------------------------------------------------------------------

                       SOLUTIONS
 total number of solutions: 18
 number of affine solutions: 18
 no multiple solutions detected

 total computation is 7.1e-02 seconds
 maximum residual is 5.4e-07
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Hilbert functions of chopped ideals!

Mathematics > Commutative Algebra

[Submitted on 5 Jul 2023]

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Fulvio Gesmundo, Leonie Kayser, Simon Telen

A chopped ideal is obtained from a homogeneous ideal by considering only the generators of a fixed degree. We investigate cases in which the chopped ideal defines the same finite set of points as the original one-dimensional ideal. The complexity of computing these points from the chopped ideal is governed by the Hilbert function and regularity. We conjecture values for these invariants and prove them in many cases. We show that our conjecture is of practical relevance for symmetric tensor

Recap

We are lead to the following setup:

- Given a general form $F = \sum_{i=1}^{r} L_i^d \in \mathbb{C}[X_1, \dots, X_n]_d$ of "small" rank r
- Decomposition is unique, want to find $Z = \{[L_1], \dots, [L_r]\} \in \mathbb{P}^n$
- Want to solve Catalecticant polynomial system $\mathcal{F}$ using the eigenvalue method
- Is $\mathcal{Z}eros(\mathcal{F}) = Z$? With(out) multiplicities?
- What is the Hilbert function of the ideal $\langle \mathcal{F} \rangle_S \subseteq S$? When $= r$?

For which forms is our algorithmic approach even possible?

- For a set of points Z consider the vanishing ideal and chopped ideal

$$I = \{\, f \in S \mid f(Z) = 0 \,\}\,, \qquad I_{\langle d \rangle} = \langle \{\, f \in S_d \mid f(Z) = 0 \,\} \rangle_S$$

- Generally $I_{\langle d \rangle} \subsetneq I$, we need $\mathcal{Z}eros(I) \overset{?}{\underset{\text{multiplicities}}{=}} \mathcal{Z}eros(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$. Then

$$\mathcal{Z}eros(I) = \mathcal{Z}eros(I_{\langle d \rangle}) \quad \Longleftrightarrow \quad r < \binom{n+d}{n} - n \text{ or } r = 1 \text{ or } (n, r, d) = (2, 4, 2).$$

When planets align

- Let $f = (f_1, \ldots, f_s)$ be homogeneous polynomials of degree d
- A relation or syzygy of f is a vector of polynomials $(g_1, \ldots, g_s) \in S$ with

$$g_1 f_1 + \cdots + g_s f_s = 0$$

- For $1 \leq i < j < s$ have Koszul syzygy $f_j \cdot f_i + (-f_i) \cdot f_j = 0$
- A syzygy syzygy is a relation $(h_1, \ldots, h_m)$ among syzygies, ...
- There are $\binom{s}{k}$ higher Koszul syzygies of "order" k

⇝ Leads to Fröberg's estimate

$$\mathrm{hf}_S(t) - \underbrace{\mathrm{hf}_S(t-d)s}_{\text{gen's of } I_d} + \underbrace{\mathrm{hf}_S(t-2d)\binom{s}{2}}_{\text{Koszul syzygies}} - \underbrace{\mathrm{hf}_S(t-3d)\binom{s}{3}}_{\text{Koszul syzygy syzygies}} \pm \ldots$$

The conjectural Hilbert function

Expected syzygy conjecture (ESC)

For a general set of $r < \binom{n+d}{n} - n$ points in $\mathbb{P}^n$ the ideal $I_{\langle d \rangle}$ has Hilbert function

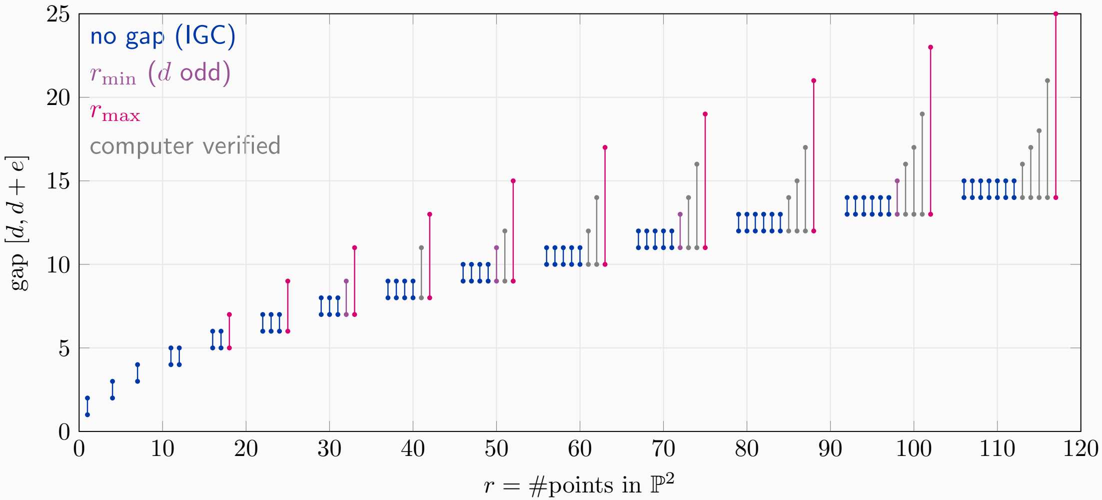
$$\mathrm{hf}_{S/I_{\langle d \rangle}}(t) = \begin{cases} \displaystyle\sum_{k \geq 0} (-1)^k \cdot \mathrm{hf}_S(t - kd) \cdot \binom{\binom{n+d}{n} - r}{k} & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the first integer $> d$ such that the sum is $\leq r$.

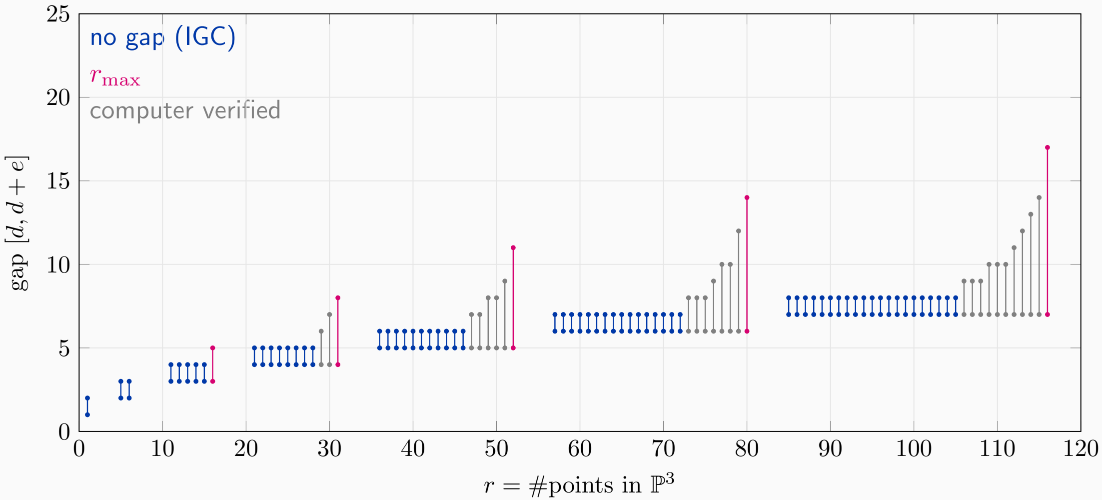
- One can extract the saturation gap length from this formula
- If $W \subseteq S_d$ is a random vector subspace of dim. $\binom{n+d}{n} - r$, then this sum is the expected Hilbert function of $S/\langle W \rangle_S$ (until sum ≤ 0)

Visualization of the saturation gaps in $\mathbb{P}^2$

▷ ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is



Visualization of the saturation gaps in $\mathbb{P}^3$



Main results

Theorem (Gesmundo, Kayser & Telen)

Conjecture (ESC) is true in the following cases:

- $r_{\max} := \binom{n+d}{n} - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\left((n+1)\binom{n+d}{n} - \binom{n+d+1}{n}\right)$ *and [*$n \leq 4$ *or* $d \gg 0$*]*
- *In a large number of individual cases in low dimension (table below).*

The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

n	2	3	4	5	6	7	8	9	10
r	≤ 1825	≤ 1534	≤ 991	≤ 600	≤ 447	≤ 316	≤ 333	≤ 204	≤ 259
d	≤ 58	≤ 18	≤ 9	≤ 6	≤ 4	≤ 3	≤ 3	≤ 2	≤ 2

Thank you! Questions?

arXiv:2307.02560

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