MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

Inverse limits of varieties and their ℓ^2-distance degree

Algebraic aspects of Metric and Integral Geometry

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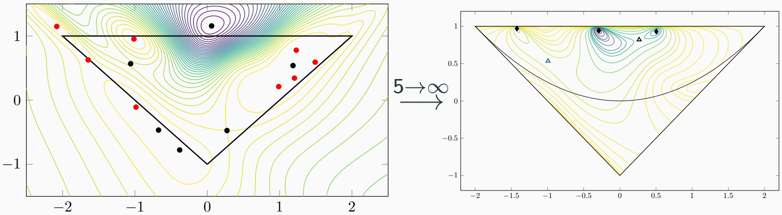


Approximate tensor decompositions

- Given a variety $X \subseteq \mathbb{C}^N$ and a point $T \in \mathbb{R}^N$, find closest point on $X(\mathbb{R})$
- Distance measured using non-degenerate quadric $Q(x) = x^\intercal \Lambda x$

Definition (Euclidean distance degree, $\mathrm{EDD}_Q(X)$)

The number of complex critical points of $y \mapsto Q(x-y)$ on X_{reg} for general $x \in \mathbb{R}^N$ is the Euclidean Distance degree of X (with respect to Q).

- For generic quadric obtain generic EDD $\geq$ “specific” $\mathrm{EDD}_Q(X)$
- Focus here on r-th secant variety of rational normal curve

$$X = \mathcal{X}_{d,r} = \widehat{\sigma}_r \nu_d(\mathbb{P}^1) = \left\{ T \in \operatorname{Sym}^d \mathbb{C}^2 \;\middle|\; \overline{\mathrm{rk}}\, T \leq r \right\} \subseteq \mathbb{C}^{d+1}$$

- $\mathcal{X}_{d,r}$ has dimension $2r$; determinantal variety of generic Hankel matrices
- $\mathrm{EDD}_Q(\mathcal{X}_{d,r})$ is algebraic degree of approximate rank-r decomposition
- Optimization problem appears in signal processing applications and system engineering

Theorem (Kayser–Lagauw 2026+a)

1. *The generic Euclidean distance degree is*

$$\mathrm{EDD}_{\mathrm{gen}}(\mathcal{X}_{d,r}) = \sum_{j=0}^{r} \binom{d-2r+j}{j} 3^j.$$

   The critical locus can be realized as a degeneracy locus of a bundle map on $\mathbb{P}^r$.

2. *We obtain an explicit criterion for a quadric* Q *to be "generic".*
3. *Implement algorithm to compute bounds on* $\mathrm{EDD}_Q(\mathcal{X}_{d,r})$ *also for non-generic* Q.
4. *Extensive computational studies lead to new conjectures on* $\mathrm{EDD}_\Theta(\mathcal{X}_{d,r})$ *for Bombieri inner product* Θ*; prove new cases (e.g.* $\mathrm{EDD}_\Theta(\mathcal{X}_{6,2}) = 28$*).*

- Reproduce old and find new results using simple approach and basic intersection theory
- Extends previous works by [Ottaviani–Spaenlehauer–Sturmfels] and [Panah]

To infinity and beyond

- Identify $\mathcal{X}_{d,r} = \widehat{\sigma}_r \nu_d(\mathbb{P}^1)$ with sequences $y \in \mathbb{C}^{d+1}$ satisfying order r linear recurrence
- Fixing $r \in \mathbb{N}$ and increasing $d \to \infty$ leads in the inverse limit to the variety $\mathcal{X}_{\infty,r}$

$$\widehat{\sigma}_r \nu_7 \mathbb{P}^1 \twoheadleftarrow \widehat{\sigma}_r \nu_8 \mathbb{P}^1 \twoheadleftarrow \widehat{\sigma}_r \nu_9 \mathbb{P}^1 \twoheadleftarrow \cdots \twoheadleftarrow \widehat{\sigma}_r \nu_\infty \mathbb{P}^1 = \{\, y \in \mathbb{C}^{\mathbb{N}} \mid y \text{ obeys order } r \text{ lin. recurrence} \,\}$$

- Naturally parametrized by rational functions [Kronecker]

$$y \mapsto Y(z) = \frac{b(z)}{a(z)} = \sum_{j=1} y_j z^{-j} \in \mathbb{C}(z), \qquad y = (y_j)_j = \big(\textstyle\sum_{a(\omega)=0} \lambda_\omega \omega^j\big)_j$$

- $\|\cdot\|_2$ on $\mathbb{R}^N$ becomes ℓ^2-norm on sequences $\|y\|_{\ell^2}^2 = \|Y\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{S}^1} |Y(z)| \frac{\mathrm{d}z}{z}$

$\rightsquigarrow$ Impose Schur-stability of $a(z)$ ($|\omega| < 1$)

- Approximation from $\mathcal{X}_{\infty,R}(\mathbb{R})$ to $\mathcal{X}_{\infty,r}(\mathbb{R})$ known as model order reduction

Walsh meets Porteous: ℓ^2-distance degree of $\mathcal{X}_{\infty,r}$

- Model order reduction problem: Given $Y(z) = \frac{d(z)}{c(z)}$ (c stable, $\deg d < \deg c \leq R$)

$$\underset{\widehat{Y}}{\text{minimize}} \ \|\widehat{Y} - Y\|^2_{\mathcal{H}_2}, \qquad \text{subject to } \widehat{Y} = \frac{b(z)}{a(z)}, \ \deg b < \deg a \leq r, \ a \text{ stable}$$

- [Walsh]: Criticality is characterized by $Y(\omega^{-1}) = \widehat{Y}(\omega^{-1})$, $Y'(\omega^{-1}) = \widehat{Y}'(\omega^{-1})$

Theorem (Lagauw–Kayser 2026+b, main result)

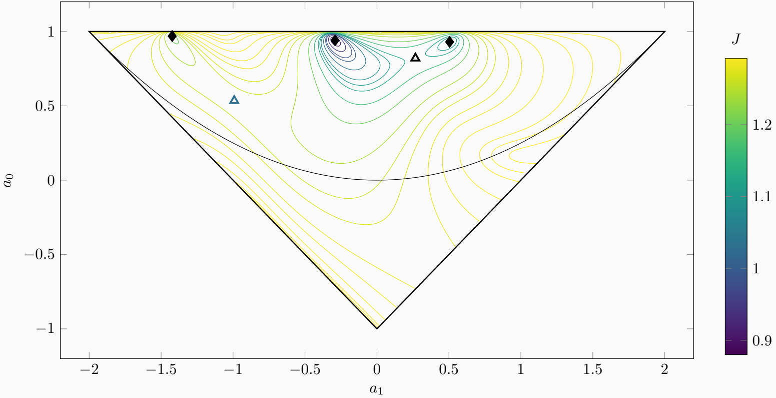
- *The critical incidence $\{(Y, \widehat{Y}) \mid \widehat{Y}$ critical point to $Y\}$ is an irreducible variety.*
- *The algebraic degree of model order reduction (order R to r) is*

$$\sum_{j=0}^{r} \binom{R-r-1+j}{j} 2^j.$$

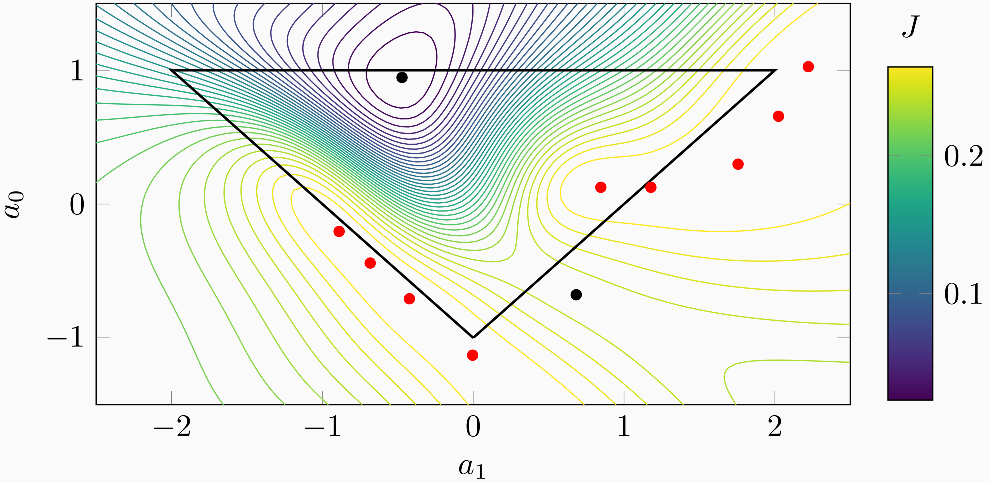
Example: Model order reduction from order 6 to 2

$$Y(z) = \frac{0.0448z^5 + 0.2368z^4 + 0.0013z^3 + 0.0211z^2 + 0.2250z + 0.0219}{z^6 - 1.2024z^5 + 2.3675z^4 - 2.0039z^3 + 2.2337z^2 - 1.0420z + 0.8513}$$

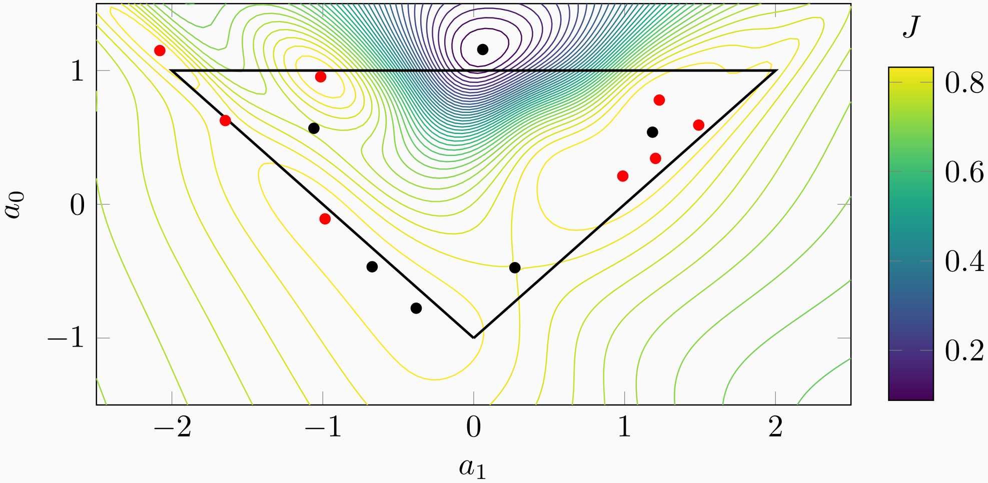
$$y = (0.0448, 0.2907, 0.2447, 0.2830, 0.2123, 0.3245, 0.0438, 0.5013, 0.4671, \dots)$$



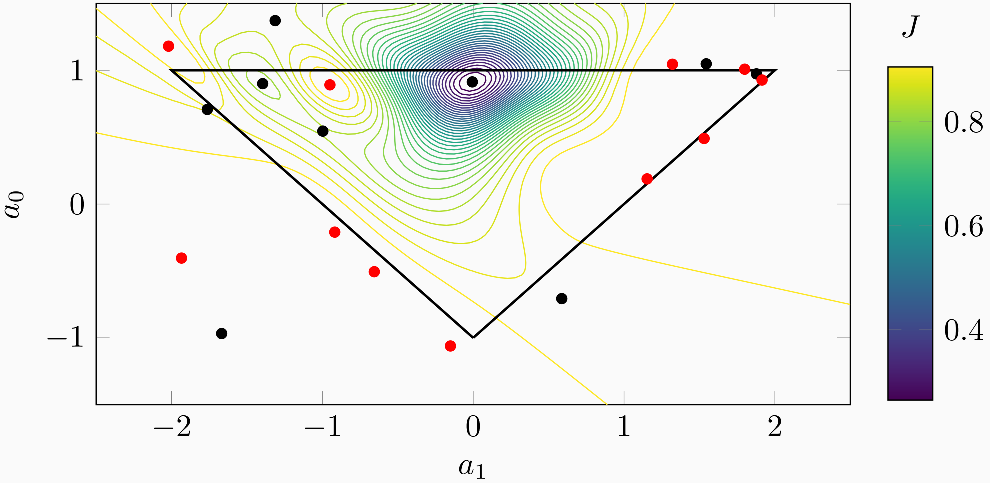
ED minimization of $(y_0, \ldots, y_4)$ to $\mathcal{X}_{4,2}$ $(N = 5)$



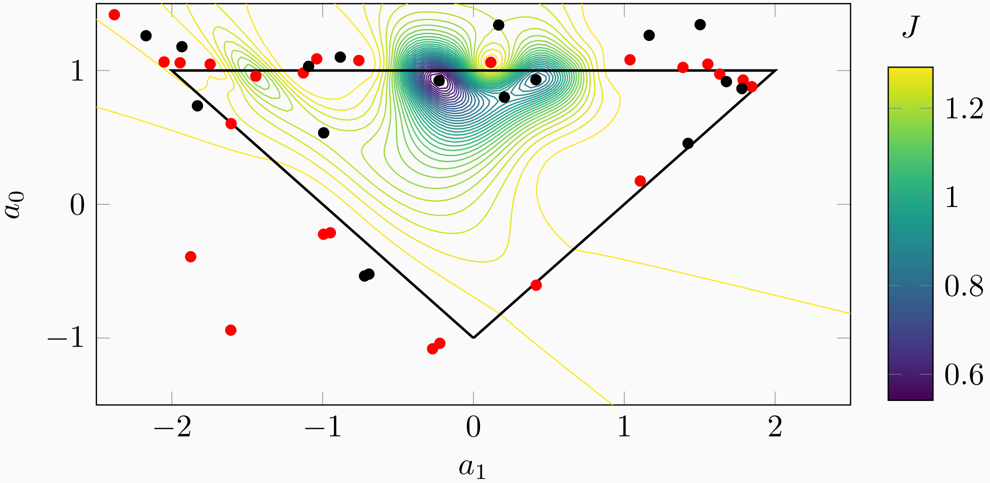
ED minimization of $(y_0, \ldots, y_9)$ to $\mathcal{X}_{9,2}$ ($N = 10$)



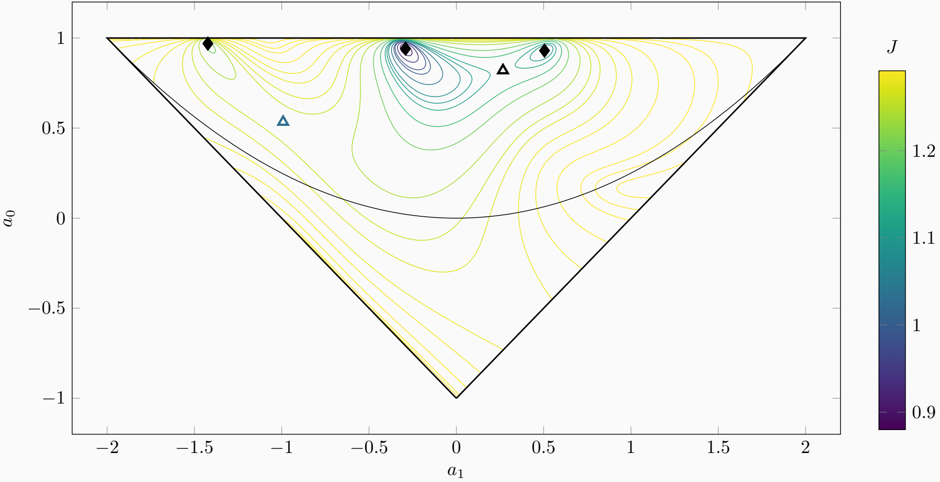
ED minimization of $(y_0, \ldots, y_{14})$ to $\mathcal{X}_{14,2}$ $(N = 15)$



ED minimization of $(y_0, \ldots, y_{29})$ to $\mathcal{X}_{29,2}$ ($N = 30$)



ℓ_2-distance minimization of y to $\mathcal{X}_{\infty,2}$



Many open questions

- What is the bidegree of the critical incidence?
- Why is the standard norm here the “correct one” and not the Bombieri norm?
- Can you formalize that critical points for finite N converge to critical points of the MOR problem as $N \to \infty$? What happens to the unstable critial points?
- Are the degeneracy loci descriptions of the two problems related?
- Are there other interesting (families of) varieties in $\mathbb{C}^\infty$ with ℓ^2-distance degree?
- Does such a phenomenon appear for other sequences of varieties as $N \to \infty$?

Thank you! Questions?