Hyperplane arrangements, reciprocal linear spaces, logarithmic discriminants and beyond

Graduate Online Combinatorics Colloquium

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES

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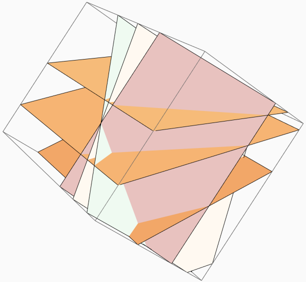
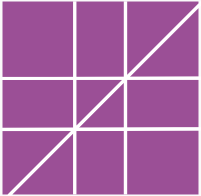


Affine hyperplane arrangements

$$\ell(x) = (x_1, x_2, x_1 - 1, x_2 - 1, x_2 - x_1)$$
$$= (1, x_1, x_2) \cdot \left[\begin{array}{ccccc} 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$



Definition (Hyperplane arrangement)

A hyperplane arrangement $\mathcal{A} \subseteq \mathbb{K}^d$ ($\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$) is a union of affine hyperplanes

$$\mathcal{A} = H_1 \cup \cdots \cup H_n = \mathbb{V}(\ell_1 \cdots \ell_n) \subseteq \mathbb{K}^d, \qquad \ell(x) = Ax + b.$$

- $A^\mathsf{T} \in \mathbb{K}^{d \times n}$ and $L := [b \mid A]^\mathsf{T} \in \mathbb{K}^{(d+1) \times n}$ are the "little" and "big" matrices of $\mathcal{A}$
- The rank of $\mathcal{A}$ is $\operatorname{rk} A$ (= #independent normals); $\mathcal{A}$ is essential if $\operatorname{rk} \mathcal{A} = d$
- X will always denote the hyperplane arrangement complement $X := \mathbb{K}^d \setminus \mathcal{A}$

Topology of arrangement complements

- The characteristic polynomial of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}\subseteq\mathcal{A},\, \bigcap\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|} \cdot t^{\dim_{\mathbb{K}} \bigcap \mathcal{B}} = \sum_{F\in \mathrm{L}(\mathcal{A})} \mu(F)\cdot t^{\dim_{\mathbb{K}} F}$$

Theorem (Topological invariants via the characteristic polynomial)

$(\mathbb{C})$ *The cohomology groups* $\mathrm{H}^i_{\mathsf{sing}}(X,\mathbb{Z})$ *are free of finite rank* $b_i(X)$ *and*

$$\sum_{i\geq 0} b_i(X)\cdot t^i \;=\; (-t)^d \chi_{\mathcal{A}}(-t^{-1}) \qquad \rightsquigarrow \qquad \chi(X) \;=\; \chi_{\mathcal{A}}(1)$$

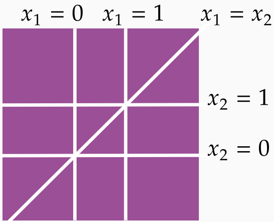
$(\mathbb{R})$ *Let* $\mathrm{ch}(\mathcal{A})$ *be the number of connected components of* $X = \mathbb{R}^d \setminus \mathcal{A}$ *and let* $\mathrm{bch}(\mathcal{A})$ *be the number of (relatively) bounded components, then*

$$\mathrm{ch}(\mathcal{A}) = (-1)^d \chi_{\mathcal{A}}(-1), \qquad \mathrm{bch}(\mathcal{A}) = (-1)^{\mathrm{rk}\,\mathcal{A}} \chi_{\mathcal{A}}(1)$$

Where are the matroids?

Previous example of $X = \mathcal{M}_{0,5}$:

$$\chi_{\mathcal{A}}(t) = t^2 - 5t + 8 - 2 + 0 - 0 = t^2 - 5t + 6$$
$$\chi(\mathbb{C}^2 \setminus \mathcal{A}_{\mathbb{C}}) = (-1)^2 \cdot \mathrm{bch}(\mathcal{A}) = \chi_{\mathcal{A}}(1) = 2$$
$$(-1)^2 \cdot \mathrm{ch}(\mathcal{A}) = \chi_{\mathcal{A}}(-1) = 12$$



- Let $e = (1, 0, \dots, 0)^{\mathsf{T}} \in \mathbb{C}^{d+1}$ ($e \mathrel{\hat{=}} H_\infty$) and set $\mathrm{c}L := [e \mid L] = \left[\begin{array}{c|c} 1 & b^{\mathsf{T}} \\ \hline 0 & A^{\mathsf{T}} \end{array}\right]$
- The affine matroid of $\mathcal{A}$ is $(M := \mathrm{M}(\mathrm{c}L), e)$
- (M, e) remembers the big ($M \backslash e = \mathrm{M}(L)$) and little ($M/e = \mathrm{M}(A^{\mathsf{T}})$) matroid and

$$\chi_{\mathcal{A}}(t) = \overline{p}_M(t) := \frac{1}{t-1} p_M(t), \qquad \chi(X) = (-1)^{\mathrm{rk}(M)-1} \beta(M)$$

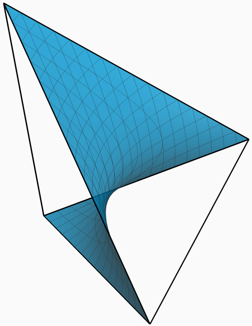
⇝ Today mostly in the bi-uniform case: Both $\mathrm{M}(L)$ and $\mathrm{M}(A^{\mathsf{T}})$ are *uniform*

Maximum Likelihood Estimation in Algebraic Statistics

- Let $X \subseteq (\mathbb{C}^\times)^n$ be a d-dimensional smooth variety
- Discrete statistical model $X \cap \Delta_{n-1} = \{\, p \in X \cap \mathbb{R}^n \mid p_i > 0,\ p_1 + \cdots + p_n = 1 \,\}$
- Given data points $u \in \mathbb{N}^n$, which parameter maximizes the log-likelihood function

$$\mathcal{L}_u(x) = \log x_1^{u_1} \cdots x_n^{u_n}, \qquad x \in X \cap \Delta_{n-1}?$$

- Critical equations: $x \in \mathrm{Crit}_X(u) := \{\, x \in X \mid \nabla \mathcal{L}_u(x) = 0 \,\}$
- $\mathrm{Crit}_X(u)$ is a *finite* set of $\mathrm{MLdeg}(X)$ non-degenerate critical points for *general* data $u \in \mathbb{N}^n$ (or $\mathbb{C}^n$)
- [Huh13] $\mathrm{MLdeg}(X) = (-1)^d \chi(X)$
- Extensively studied for toric models (exponential families), linear models, determinantal varieties, ...



Linear models and scattering amplitudes

- Let $\mathcal{A} \subseteq \mathbb{C}^d$ be an arrangement of n hyperplanes defined by $\ell(x) = Ax + b$
- $X = \mathbb{C}^d \setminus \mathcal{A}$ is a very affine variety if and only if $\mathcal{A}$ is essential ($\operatorname{rk} \mathcal{A} = d$)
- In this case parametrizes linear model $\ell\colon X \hookrightarrow (\mathbb{C}^\times)^n$ can assume $\sum_j \ell_j = 1$
- Log-likelihood function or master function given by

$$\mathcal{L}_u(x) = u_1 \log \ell_1(x) + \cdots + u_n \log \ell_n(x), \qquad \nabla \mathcal{L}_u(x) = A^\mathsf{T} \operatorname{diag}(1/\ell_1, \ldots, 1/\ell_n) u$$

- Varchenko: For real arrangements, $\operatorname{Crit}_X(u) \leftrightarrow$ bounded chambers of $\mathbb{R}^d \cap \mathcal{A}$
- Critical equations appear as scattering equations in bi-adjoint scalar ϕ^3-theories (Cachazo, He & Yuan [CHY14])

What is “general” data?

- Moving from general to special $u \in \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$, what can happen to $\mathrm{Crit}_X(u)$?
  1. Two critical points collide to form a non-reduced/degenerate point
  2. A positive-dimensional component appears
  3. A critical point disappears to infinity
- Outside of 1.-3. the finite set of critical points has constant size $\mathrm{MLdeg}(X)$
- The closure of 1.-3. was called the *data discriminant* by Rodriguez & Tang [RT15]
- 3. was studied by Sattelberger & van der Veer [SvdV23]

Definition (Ad-hoc definition of $\nabla_{\log}(X)$)

The *logarithmic discriminant* of a (smooth) variety $X \hookrightarrow (\mathbb{C}^\times)^n$ is

$$\nabla_{\log}(X) := \overline{\{\, u \in \mathbb{P}^{n-1} \mid \mathrm{Crit}_X(u) \text{ is infinite or non-reduced} \,\}}.$$

⇝ Goal: Understand logarithmic discriminants of hyperplane arrangements!

Three points enter a bar

- Three points on a line $\mathcal{A} = \{0, 1, b\} \subseteq \mathbb{C}^1$
- Model is a line $X = \mathbb{C}^1 \setminus \mathcal{A} \hookrightarrow (\mathbb{C}^\times)^3$ parametrized by $(x, x-1, x-b)$,

$$\mathcal{L}_u(x) = u_1 \log x + u_2 \log(x-1) + u_3 \log(x-b)$$

- Single critical equation in $x \in \mathbb{C}^1 \setminus \mathcal{A}$

$$\frac{u_1}{x} + \frac{u_2}{x-1} + \frac{u_3}{x-b} = 0 \quad \Longleftrightarrow \quad u_1(x-1)(x-b) + u_2 x(x-b) + u_3 x(x-1) = 0$$

- When does this quadric in x have a double root? Highschool discriminant vanishes!

$$\Delta_{\log}(X) = (b-1)^2\, u_1^2 + 2b(b-1)\, u_1 u_2 + b^2\, u_2^2 - 2(b-1)\, u_1 u_3 + 2b\, u_2 u_3 + u_3^2$$

- $\Delta_{\log}(X)$ itself is a smooth quadric in u with discriminant $-4b^2(b-1)^2$

Ramification and its consequences

- Let $f\colon V \to W$ be a dominant map of smooth irreducible varieties of dimension n
- The ramification locus $\operatorname{Ram}(f) \subseteq V$ is the hypersurface

$$\operatorname{Ram}(f) = \{\, x \in V \mid x \in f^{-1}(f(x)) \text{ is not isolated or reduced} \,\} = \mathbb{V}(\det J_f(x))$$

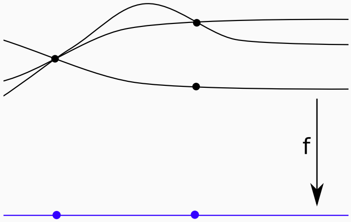
- The branch locus is the image closure $\operatorname{Branch}(f) = \overline{f(\operatorname{Ram}(f))} \subseteq W$
- Apply this to the likelihood correspondence $\mathcal{L}_X^\circ$ and the projection

$$f\colon \mathcal{L}_X^\circ := \{\, (u, x) \in \mathbb{P}^{n-1} \times X \mid \nabla \mathcal{L}_u(x) = 0 \,\} \to \mathbb{P}^{n-1}$$

Definition (True definition of $\nabla_{\log}(X)$)

The logarithmic discriminant is the branch locus of f. The ramification locus is defined in $\mathbb{P}^{n-1} \times X$ by

$$\nabla \mathcal{L}_u(x) = 0, \quad \det \operatorname{Hess}_x(\mathcal{L}_u(x)) = 0.$$



The ramification locus of a linear model

- $X = \mathbb{C}^d \setminus \mathbb{V}(\ell_1 \cdots \ell_n), \quad (\ell_1(x), \ldots, \ell_n(x))^\mathsf{T} = Ax + b$
- Here the equations of the ramification locus have a very concrete form

$$\nabla \mathcal{L}_u(x) = A^\mathsf{T} \cdot \mathrm{diag}(1/\ell_1, \ldots, 1/\ell_n) \cdot u = 0$$

$$h = \det\left(A^\mathsf{T} \cdot \mathrm{diag}\left(\frac{u_1}{\ell_1^2}, \ldots, \frac{u_n}{\ell_n^2}\right) \cdot A\right) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = d}} |A_I|^2 \frac{u^I}{(\ell^I)^2}$$

- Critical equations are linear in the u_j $\rightsquigarrow$ substitute $u_1, \ldots, u_d$ in h to obtain

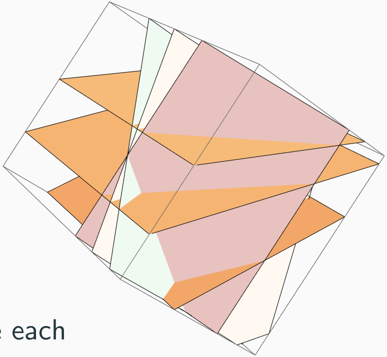
$$\tilde{h} \in \mathbb{C}[u_{d+1}, \ldots, u_n; x], \qquad \mathrm{Ram}(f) \cong \mathbb{V}(\tilde{h}) \subseteq \mathbb{P}^{n-d} \times X$$

- $\mathrm{Ram}(f)$ is an irreducible variety iff $\tilde{h}$ is irreducible in $\mathbb{C}[u_{d+1}, \ldots, u_n, x][\ell^{-1}]$!

Theorem (Irreducibility of $\mathrm{Ram}(f)$)

If the arrangement contains a subset of $d+2$ hyperplanes which is bi-uniform, then $\mathrm{Ram}(f)$ and hence $\nabla_{\log}(X)$ are irreducible varieties.

A split discriminant!



- Consider the arrangement $\mathcal{A}$ of six planes

$$\ell = (1, x_1, x_2, x_3) \cdot \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 0 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

- The first and the last three planes intersect in a line each
- The logarithmic discriminant decomposes as

$$\begin{aligned} \nabla_{\log}(X) = \ & \mathbb{V}(144u_1^2 + 120u_1u_2 + 168u_1u_3 + 25u_2^2 - 70u_2u_3 + 49u_3^2) \\ & \cup \mathbb{V}(u_4^2 - 2u_4u_5 + 4u_4u_6 + u_5^2 + 4u_5u_6 + 4u_6^2) \\ & \cup \mathbb{V}(u_1 + u_2 + u_3,\ u_4 + u_5 + u_6). \end{aligned}$$

A complete answer in $\mathbb{C}^1$

Theorem

Let $\mathcal{A} \subseteq \mathbb{C}^1$ be an arrangement of $n \geq 3$ distinct points.

1. *The ramification locus is a smooth irreducible hypersurface in $\mathcal{L}_X^\circ \subset \mathbb{P}^{n-1} \times (\mathbb{C}^1 \setminus \mathcal{A})$.*
2. *Its class in the Chow ring $\mathrm{A}^\bullet(\mathbb{P}^{n-1} \times \mathbb{P}^1) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^n, \beta^2 \rangle$ is $\alpha^2 + 2(n-2)\alpha\beta$.*
3. *The projection $f \colon \mathrm{Ram}(f) \to \nabla_{\log}(X)$ is generically bijective.*
4. *$\nabla_{\log}(X) \subseteq \mathbb{P}^n$ is an irreducible hypersurface of degree $2(n-2)$.*

- Explicit formula for defining polynomial ($\mathcal{A} = \{b_1, \ldots, b_n\}$)

$$\Delta_{\log}(X) = \mathrm{Disc}_x \left(\sum_{i=1}^{n} u_i \prod_{k \neq i} (x - b_k) \right)$$

- For $n = 4$ points $\nabla_{\log}(X) \subseteq \mathbb{P}^3$ is always a singular surface of degree 4

Reciprocal linear spaces

Definition (Reciprocal linear space)

The reciprocal linear space $\mathcal{R}_L$ of $X = \mathbb{C}^d \setminus \mathcal{A}$ is the image closure under the map

$$\gamma\colon \mathbb{C}^d \setminus \mathcal{A} \to \mathbb{P}^{n-1}, \qquad x \mapsto (1/\ell_1(x), \ldots, \ell_n(x)).$$

- Obtained by feeding the image of $L^\mathsf{T}\colon \mathbb{C}^d \to \mathbb{C}^n$ through the coordinate-wise inverse and taking closure (in $\mathbb{P}^{n-1}$)

Theorem (Proudfoot–Speyer 2006)

- $\mathcal{R}_L \subseteq \mathbb{P}^{n-1}$ *is a projectively Cohen–Macaulay variety of dimension* $d-1$*.*
- *A universal Gröbner bases of* $I(\mathcal{R}_L) \subseteq \mathbb{C}[y_1, \ldots, y_n]$ *is indexed by circuits* C *of* $\mathrm{M}(L)$*:* $f_C = \sum_{c\in C} l_c y^{C\setminus\{c\}}$, l *the relation of the columns of* L *indexed by* C*.*
- $\deg \mathcal{R}_L = |\mu(\mathrm{M}(L))| = (-1)^{\mathrm{rk}\,L} \cdot p_{\mathrm{M}(L)}(0)$

Special sections of reciprocal linear space

- Critical equations $A^{\mathsf{T}}\operatorname{diag}(1/\ell_1(x),\ldots,1/\ell_n(x))\cdot u \overset{!}{=} 0$ are non-linear on $\mathbb{C}^d \setminus X$
- However, applying the inversion $\gamma(x) = (1/\ell_1(x),\ldots,1/\ell_n(x))$, we have

$$x \in \operatorname{Crit}_X(u) \qquad \text{if and only if} \qquad \gamma(x) \in \varphi(u) := \operatorname{Ker}(A^{\mathsf{T}}\operatorname{diag}(u_1,\ldots,u_n))$$

⇝ Critical points correspond to a subset of the intersection points $\mathcal{R}_L \cap \varphi(u)$!

- x is a ramification point iff $\varphi(u)$ intersects $\mathcal{R}_L$ at $\gamma(x)$ non-transversally
- Linear space intersecting $(\mathcal{R}_L)_{\mathrm{sm}}$ non-transversally constitute the first associated hypersurface $\mathcal{Z}_1(\mathcal{R}_L) \subseteq \operatorname{Gr}(n-d,\mathbb{C}^n)$ defined by the Hurwitz form $\operatorname{Hu}_{\mathcal{R}_L} \in S(\operatorname{Gr})$

Theorem (Sturmfels 2017)

$$\deg \operatorname{Hu}_{\mathcal{R}_L} = (-1)^{\operatorname{rk} L} \cdot 2\big((\operatorname{rk} L)\cdot p_{\mathrm{M}(L)}(0) + p'_{\mathrm{M}(L)}(0)\big)$$

The Hurwitz Discriminant and general arrangements

Theorem

Let $\mathcal{A}$ be a bi-uniform arrangement of $n \geq d+2$ hyperplanes in $\mathbb{C}^d$.

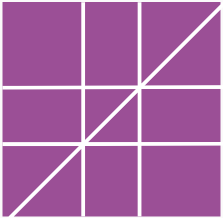
1. $\nabla_{\log}(X)$ *is an irreducible and reduced hypersurface.*
2. $\nabla_{\mathrm{Hu}}(X)$ *is a hypersurface of degree* $2d\binom{n-2}{d}$ *with full Newton polytope*
3. $\nabla_{\log}(X) \subseteq \nabla_{\mathrm{Hu}}(X)$ *coincide as sets, so* $\Delta_{\mathrm{Hu}}(X) = \Delta_{\log}(X)^e$ *for some* $e \geq 1$.
4. *If the arrangement is defined by real affine linear forms, then* $\nabla_{\log} \cap \mathbb{R}^n_+ = \emptyset$.

- Main tool: Hurwitz discriminant $\nabla_{\mathrm{Hu}}(X) \supseteq \nabla_{\log}(X)$
  1. Reciprocal linear space $\mathcal{R} := \overline{\mathrm{Im}(\ell_1^{-1} : \cdots : \ell_n^{-1})} \subseteq \mathbb{P}^{n-1}$
  2. Hurwitz form $\mathcal{Z}_1(\mathcal{R}_L) \subseteq \mathrm{Gr}(n-d, \mathbb{C}^n)$, $\deg \mathrm{Hu}_{\mathcal{R}_L} = 2(n-1-d)\binom{n-1}{d-1}$
  3. $\nabla_{\mathrm{Hu}} := \varphi^{-1}(\mathcal{Z}_1(\mathcal{R}_L))$, pullback along $\varphi\colon \mathbb{P}^{n-1} \dashrightarrow \mathrm{Gr}(n-d, \mathbb{C}^n)$, $u \mapsto \mathrm{Ker}(A^{\mathsf{T}} \operatorname{diag}(u))$

The discriminant of $\mathcal{M}_{0,m}$

- $\mathcal{M}_{0,m}$ parametrizes tuples of m distinct points on $\mathbb{P}^1$
- Fixing $(0, 1, x_1, \dots, x_{m-3}, \infty)$, it can be realized in $\mathbb{C}^{m-3}$ as the complement of the (non-const.) minors of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & x_{m-3} & 1 \end{bmatrix}$$



- Mandelstam variables s_{ij} corresponding to minor (i, j)
- Discriminant for $m = 5$ has degree $4 < 2 \cdot 2 \cdot \binom{5-2}{2} = 12$

$$\Delta_{\log}(\mathcal{M}_{0,5}) = (s_{13}s_{24} + s_{13}s_{34} + s_{14}s_{34} + s_{14}s_{23} + s_{23}s_{34} + s_{24}s_{34} + s_{34}^2)^2 - 4s_{13}s_{14}s_{23}s_{24}$$

- The Hurwitz discriminant has the extra factors

$$\Delta_{\mathrm{Hu}}(\mathcal{M}_{0,5}) = (s_{13} + s_{23} + s_{34})^2 \cdot (s_{14} + s_{24} + s_{34})^2 \cdot \Delta_{\log}(\mathcal{M}_{0,5})$$

- Conjecturally rich nested structure, degrees of $\nabla_{\log}(\mathcal{M}_{0,m})$ are $4, 30, 208, 1540, \dots$

Beyond hyperplane arrangements

- Let $f_1, \dots, f_n \in \mathbb{C}[x]$ be polynomials parametrizing a model

$$X \cong \mathbb{C}^d \setminus \mathbb{V}(f_1 \cdots f_n) \hookrightarrow (\mathbb{C}^\times)^n, \qquad x \mapsto (f_1(x), \dots, f_n(x))$$

- Case $(f, x_1, \dots, x_d)$ closely related to toric models
- $d = 1$: $\nabla_{\log}(X)$ is an irreducible hypersurface of degree $2(\# \mathbb{V}(f_1 \cdots f_n) - 2)$
- Consider a family of conics $X_z \subset \mathbb{A}^2_{\mathbb{C}[z]}$ degenerating to two lines as $z \to 0$

$$f_0 = (x_1 + x_2 + 1)(-x_1 + x_2 - 2) + z, \quad f_1 = x_1, \quad f_2 = x_2$$

- X_0 is a bi-uniform arrangement of 4 lines, hence $\deg \nabla_{\log}(X_0) = 2 \cdot 2 \cdot \binom{4-2}{2} = 4$
- $\nabla_{\log}(X_z)$ has degree 6, $\Delta_{\log}(X_z)|_{z=0} = u_0^2 \cdot \Delta_{\log}(X_0)$
- The discriminant is factor of the tact invariant: $\Delta_{\log}(X_z) = \frac{1}{u_0^6} \cdot \mathrm{tact}_x(\partial_{x_1} \mathcal{L}_u, \partial_{x_2} \mathcal{L}_u)$

Many open questions - even for linear models

- Missing piece of the puzzle: Is ∇_{Hu} reduced for a bi-uniform arrangement?
- (When) is the projection $\mathrm{Ram}(f) \to \nabla_{\log}$ generically one-to-one? …bijective?
- Is there any arrangement such that $\nabla_{\log}(\mathcal{A})$ is *not* reduced?
- Is there an arrangement of *lines* whose $\nabla_{\log}(\mathcal{A})$ is reducible?
- Is the degree of $\nabla_{\log}(\mathcal{A})$ an invariant of the matroids?
- What is the meaning of the components $\nabla_{\mathrm{Hu}} \setminus \nabla_{\log}$?
- What is the structure of the other intersection points $\varphi(u) \cap \mathcal{R}_L$?
  - Related to flats at infinity of the affine matroid (M, e)
  - In [Betti–Borovic–Telen 2025], they investigated stratification given by such flats, revealing interesting multiplicity structure

Thank you! Questions?

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