Logarithmic Discriminants of Hyperplane Arrangements

Women in Algebra and Symbolic Computation III

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

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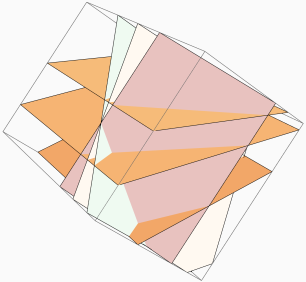


Duolingo Algebraic Geometry

- $g_1, \ldots, g_s \in \mathbb{C}[x_0, \ldots, x_n]$ polynomials, then

$$X = \mathbb{V}(g_1, \ldots, g_s) := \{\, x \in \mathbb{C}^{n+1} \mid g_1(x) = \cdots = g_s(x) = 0 \,\}$$

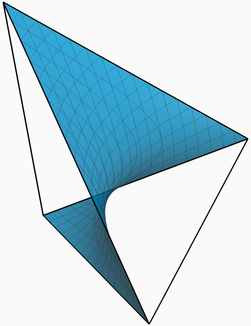
- X is called an (affine) algebraic variety
- X is irreducible if it is not a proper union of varieties
- $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus 0)/\sim$, $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{C}^\times := \mathbb{C} \setminus 0$, home of homogeneous polynomials
- Hypersurface $=$ variety defined by a single equation h, degree of $X = \deg h$
- X smooth if $J_g(x)$ has maximal rank for all $x \in X$
- x is a non-reduced/degenerate solution to $g_1, \ldots, g_s$ if the Jacobi matrix $J_g(x) = \left[\frac{\partial g_i}{\partial x_j}\right]$ does not have maximal rank

Maximum Likelihood Estimation in Algebraic Statistics

- Let $X \subseteq (\mathbb{C}^\times)^{n+1}$ be a d-dimensional smooth variety
- Discrete statistical model $X \cap \Delta_n = \{\, p \in X \cap \mathbb{R}^{n+1} \mid p_i > 0,\ p_0 + \cdots + p_n = 1 \,\}$
- Given data points $u \in \mathbb{N}^{n+1}$, which parameter maximizes the log-likelihood function

$$\mathcal{L}_u(x) = \log x_0^{u_0} \cdots x_n^{u_n}, \qquad x \in X \cap \Delta_n?$$

- Critical equations: $x \in \mathrm{Crit}_X(u) \coloneqq \{\, x \in X \mid \nabla \mathcal{L}_u(x) = 0 \,\}$
- $\mathrm{Crit}_X(u)$ is a *finite* set of $\mathrm{MLdeg}(X)$ non-degenerate critical points for *general* data $u \in \mathbb{N}^{n+1}$ (or $\mathbb{C}^{n+1}$)
- [Huh13] $\mathrm{MLdeg}(X) = (-1)^d \cdot \chi(X)$
- Extensively studied for toric models (exponential families), linear models, determinantal varieties, ...



Linear models and scattering amplitudes

- Let $\mathcal{A}$ be an essential arrangement of $n+1$ hyperplanes in $\mathbb{C}^d$

$$\mathcal{A} = \mathbb{V}(\ell_0) \cup \cdots \cup \mathbb{V}(\ell_n) \subseteq \mathbb{C}^d, \qquad (\ell_0(x), \ldots, \ell_n(x))^{\mathsf{T}} = Ax + b, \quad L^{\mathsf{T}} = [b \mid A]$$

- Parametrizes linear model $X := \mathbb{C}^d \setminus \mathcal{A} \overset{\ell}{\hookrightarrow} (\mathbb{C}^\times)^{n+1}$, can assume $\sum_j \ell_j = 1$
- Log-likelihood function or master function given by

$$\mathcal{L}_u(x) = u_0 \log \ell_0(x) + \cdots + u_n \log \ell_n(x), \qquad \nabla \mathcal{L}_u(x) = A^{\mathsf{T}} \operatorname{diag}(1/\ell_0, \ldots, 1/\ell_n) u$$

- If the ℓ_j are real, then $\mathrm{MLdeg}(X)$ is the number of bounded chambers of $\mathcal{A} \cap \mathbb{R}^d$
- Critical equations appear as scattering equations in bi-adjoint scalar ϕ^3-theories (Cachazo, He & Yuan [CHY14])

What is "general" data?

- Moving from general to special $u \in \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$, what can happen to $\mathrm{Crit}_X(u)$?
  1. Two critical points collide to form a non-reduced/degenerate point
  2. A positive-dimensional component appears
  3. A critical point disappears to infinity
- The closure of 1.-3. was called the *data discriminant* by Rodriguez & Tang [RT15]
- 3. was studied by Sattelberger & van der Veer [SvdV23]

Definition (Ad-hoc definition of $\nabla_{\log}(X)$)

The *logarithmic discriminant* of a (smooth) variety $X \hookrightarrow (\mathbb{C}^\times)^{n+1}$ is

$$\nabla_{\log}(X) := \overline{\{\, u \in \mathbb{P}^n \mid \mathrm{Crit}_X(u) \text{ is infinite or non-reduced} \,\}}.$$

⇝ Goal: Understand logarithmic discriminants of hyperplane arrangements!

Three points enter a bar

- Three points on a line $\mathcal{A} = \mathbb{V}(x\,(x+1)\,(x+b)) = \{0, -1, -b\} \subseteq \mathbb{C}^1$ $(b \notin \{0,1\})$
- Model is a line $X \subseteq (\mathbb{C}^\times)^3$ parametrized by $(x, x+1, x+b)$,

$$\mathcal{L}_u(x) = u_0 \log x + u_1 \log(x+1) + u_2 \log(x+b)$$

- A single critical equation in $x \in \mathbb{C}^1 \setminus \mathcal{A}$

$$\frac{u_0}{x} + \frac{u_1}{x+1} + \frac{u_2}{x+b} = 0 \quad \Longleftrightarrow \quad u_0(x+1)(x+b) + u_1 x(x+b) + u_2 x(x+1) = 0$$

- When does this quadric in x have a double root? A: Highschool discriminant vanishes!

$$\Delta_{\log}(X) = (b-1)^2\, u_0^2 + 2b(b-1)\, u_0 u_1 + b^2\, u_1^2 - 2(b-1)\, u_0 u_2 + 2b\, u_1 u_2 + u_2^2$$

- $\Delta_{\log}(X)$ itself is a smooth quadric in u with discriminant $-4b^2(b-1)^2$

Ramification and its consequences

- Let $f\colon V \to W$ be a dominant map of smooth irreducible varieties of dimension n
- The ramification locus $\operatorname{Ram}(f) \subseteq V$ is the hypersurface

$$\operatorname{Ram}(f) = \left\{\, x \in V \;\middle|\; x \in f^{-1}(f(x)) \text{ is not isolated or reduced} \,\right\} = \mathbb{V}(\det J_f(x))$$

- The branch locus is the image closure $\operatorname{Branch}(f) = \overline{f(\operatorname{Ram}(f))} \subseteq W$
- Apply this to the likelihood correspondence

$$f\colon \mathcal{L}_X^\circ = \{\,(u,x) \in \mathbb{P}^n \times X \mid \nabla \mathcal{L}_u(x) = 0\,\} \to \mathbb{P}^n$$

Definition (True definition of $\nabla_{\log}(X)$)

The logarithmic discriminant is the branch locus of the projection f. The ramification locus is defined in $\mathbb{P}^n \times X$ by

$$\nabla \mathcal{L}_u(x) = 0, \qquad \det \operatorname{Hess}_x(\mathcal{L}_u(x)) = 0.$$

Irreducibility of $\mathrm{Ram}(f)$

- $X = \mathbb{C}^d \setminus \mathbb{V}(\ell_0 \cdots \ell_n), \quad (\ell_0(x), \ldots, \ell_n(x))^\mathsf{T} = Ax + b$
- Here the equations of the ramification locus have a very concrete form

$$\nabla \mathcal{L}_u(x) = A^\mathsf{T} \cdot \mathrm{diag}(1/\ell_0, \ldots, 1/\ell_n) \cdot u = 0$$

$$h = \det\left(A^\mathsf{T} \cdot \mathrm{diag}\left(\frac{u_0}{\ell_0^2}, \ldots, \frac{u_n}{\ell_n^2}\right) \cdot A\right) = \sum_{\substack{I \subseteq \{0,\ldots,n\} \\ |I|=d}} |A_I|^2 \frac{u^I}{(\ell^I)^2}$$

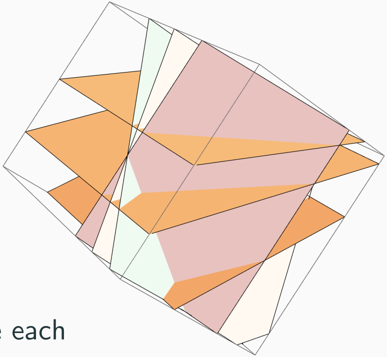
- Critical equations are linear in the u_j $\rightsquigarrow$ substitute them in h to obtain

$$\tilde{h} \in \mathbb{C}[u_d, \ldots, u_n; x], \qquad \mathrm{Ram}(f) \cong \mathbb{V}(\tilde{h}) \subseteq \mathbb{P}^{n-d} \times X$$

Theorem

If the arrangement contains a subset of $d+2$ hyperplanes which is bi-uniform (to be defined in a moment), then $\mathrm{Ram}(f)$ and hence $\nabla_{\log}(X)$ are irreducible varieties.

A split discriminant!



- Consider the arrangement $\mathcal{A}$ of six planes

$$\ell = (1, x_1, x_2, x_3) \cdot \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 0 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

- The first and the last three planes intersect in a line each
- The logarithmic discriminant decomposes as

$$\begin{aligned} \nabla_{\log}(X) = \ & \mathbb{V}(144u_0^2 + 120u_0u_1 + 168u_0u_2 + 25u_1^2 - 70u_1u_2 + 49u_2^2) \\ & \cup \mathbb{V}(u_3^2 - 2u_3u_4 + 4u_3u_5 + u_4^2 + 4u_4u_5 + 4u_5^2) \\ & \cup \mathbb{V}(u_0 + u_1 + u_2, u_3 + u_4 + u_5). \end{aligned}$$

A complete answer in $\mathbb{C}^1$

Theorem

Let $\mathcal{A} \subseteq \mathbb{C}^1$ be an arrangement of $n+1 \geq 3$ distinct points.

1. *The ramification locus is a smooth irreducible hypersurface in $\mathbb{P}^n \times (\mathbb{C}^1 \setminus \mathcal{A})$.*
2. *The class of $\overline{\mathrm{Ram}(f)}$ in the cohomology ring $\mathrm{H}^{2\bullet}_{\mathsf{sing}}(\mathbb{P}^n \times \mathbb{P}^1) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^{n+1}, \beta^2 \rangle$ is $\alpha^2 + 2(n-1)\alpha\beta$.*
3. *The projection $f\colon \mathrm{Ram}(f) \to \nabla_{\log}(X)$ is generically bijective.*
4. *$\nabla_{\log}(X) \subseteq \mathbb{P}^n$ is an irreducible hypersurface of degree $2(n-1)$.*

- Explicit formula for defining polynomial

$$\Delta_{\log}(X) = \mathrm{Disc}_x\left(\sum_{i=0}^{n} u_i \prod_{k \neq i}(x + b_k)\right)$$

- For $n+1 = 4$ points $\nabla_{\log}(X) \subseteq \mathbb{P}^3$ is always a singular surface of degree 4

General arrangements

- $\mathcal{A}$ defined by $(\ell_0(x), \ldots, \ell_n(x))^\mathsf{T} = Ax + b$
- $k \times n$ matrix is uniform if all sets of k columns are linearly independent
- *Little matroid* $\mathrm{M}(A^\mathsf{T})$, *big matroid* $\mathrm{M}([b|A]^\mathsf{T})$, bi-uniform $=$ both are uniform
- Hurwitz discriminant $\nabla_{\mathrm{Hu}}(X) \supseteq \nabla_{\log}(X)$ is a second kind of discriminant

Theorem

Let $\mathcal{A}$ be a bi-uniform arrangement of $n+1 \geq d+2$ hyperplanes in $\mathbb{C}^d$.

1. $\nabla_{\mathrm{Hu}}(X)$ *is a hypersurface of degree* $2d\binom{n-1}{d}$ *with full Newton polytope*
2. $\nabla_{\log}(X)$ *is an irreducible and reduced hypersurface.*
3. $\nabla_{\log}(X) \subseteq \nabla_{\mathrm{Hu}}(X)$ *coincide as sets, so* $\nabla_{\mathrm{Hu}}(X) = \mathbb{V}(\Delta_{\log}(X)^e)$ *for some* $e \geq 1$.
4. *If the arrangement is defined by real affine linear forms, then* $\nabla_{\log} \cap \mathbb{R}_+^{n+1} = \emptyset$.

- We know that equality holds for $d = 1$, expect this to always hold true

Many open questions

- Missing piece of the puzzle: Is ∇_{Hu} reduced for a bi-uniform arrangement?
- (When) is the projection $\mathrm{Ram}(f) \to \nabla_{\log}$ generically one-to-one? …bijective?
- Is there any arrangement such that $\nabla_{\log}(\mathcal{A})$ is *not* reduced?
- Is there an arrangement of *lines* whose $\nabla_{\log}(\mathcal{A})$ is reducible?
- Is the degree of $\nabla_{\log}(\mathcal{A})$ an invariant of the little and big matroid?
- What is the meaning of the components $\nabla_{\mathrm{Hu}} \setminus \nabla_{\log}$?

Thank you! Questions?

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The discriminant of $\mathcal{M}_{0,m}$

- $\mathcal{M}_{0,m}$ parametrizes tuples of m points on the projective line $\mathbb{P}^1$
- By fixing $(0, 1, x_1, \ldots, x_{m-3}, \infty)$, it can be realized as the complement in $\mathbb{C}^{m-3}$ of the $n = \binom{m-1}{2} - 1$ minors of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & x_{m-3} & 1 \end{pmatrix}$$

- Variable corresponding to minor (i, j) are Mandelstam invariants s_{ij}
- Discriminant for $m = 5$ has degree $4 < 2 \cdot 2 \cdot \binom{5-2}{2} = 12$

$$\Delta_{\log}(\mathcal{M}_{0,5}) = (s_{13}s_{24} + s_{13}s_{34} + s_{14}s_{34} + s_{14}s_{23} + s_{23}s_{34} + s_{24}s_{34} + s_{34}^2)^2 - 4s_{13}s_{14}s_{23}s_{24}$$

- The Hurwitz discriminant has the extra factors

$$\Delta_{\mathrm{Hu}}(\mathcal{M}_{0,5}) = (s_{13} + s_{23} + s_{34})^2 \cdot (s_{14} + s_{24} + s_{34})^2 \cdot \Delta_{\log}(\mathcal{M}_{0,5})$$

- Conjecturally rich nested structure, degrees of $\nabla_{\log}(\mathcal{M}_{0,m})$ are $4, 30, 208, 1540, \ldots$

QUEER IN MATH DAY

June 11, 2025 www.mis.mpg.de/qimd

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