Mukai Lifting of self-dual points in $\mathbb{P}^6$

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Mathematics > Algebraic Geometry

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Mukai lifting of self-dual points in $\mathbb{P}^6$

Barbara Betti, Leonie Kayser

A set of $2n$ points in $\mathbb{P}^{n-1}$ is self-dual if it is invariant under the Gale tr
on canonical curves, Petrakiev showed that a general self-dual set of 14
intersection of the Grassmannian $\mathrm{Gr}(2,6)$ in its Plücker embedding in $\mathbb{P}$
In this paper we focus on the inverse problem of recovering such a linea
dual set of points. We use numerical homotopy continuation to approac
algorithm in Julia to solve it. Along the way we also implement the forw

Mukai Grassmannians

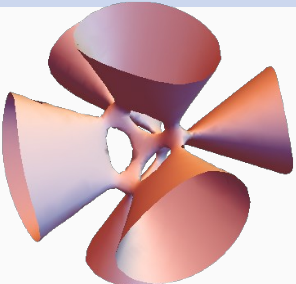
Theorem (Shigeru Mukai (1987))

For a genus $g \geq 6$, a sufficiently general

- *canonical curve ($g \leq 9$),*
- *pseudo-polarized K3 surface ($g \leq 10$), or*
- *prime Fano 3-fold ($g \leq 10$, $g = 12$)*

is a "linear section" of a homogeneous variety $X_g \subseteq \mathbb{P}V$.

g	V	$X_g \subseteq \mathbb{P}V$	$\dim X_g$	$\dim \mathbb{P}V$
6	$\bigwedge^2 \mathbb{C}^5$	$\mathrm{Gr}(2, \mathbb{C}^5)$	6	9
7	$\bigwedge^{\text{even}} \mathbb{C}^{10}$	$\mathrm{LG}_+(5, \mathbb{C}^{10})$	10	15
8	$\bigwedge^2 \mathbb{C}^6$	$\mathrm{Gr}(2, \mathbb{C}^6)$	8	14
9	$\bigwedge^3 \mathbb{C}^6 / \omega \wedge \mathbb{C}^6$	$\mathrm{Gr}_\omega(3, \mathbb{C}^6)$	6	13



Self-dual points

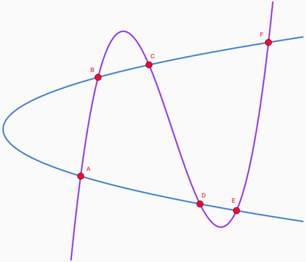
Let $\Gamma \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ be a set of $2n$ non-degenerate points identified with $\Gamma \in \mathbb{C}^{n\times 2n}$. **TFAE:**

1. $\exists \Lambda \in \mathrm{Diag}(2n)$ invertible such that $\Gamma \cdot \Lambda \cdot \Gamma^{\mathsf{T}} = \mathbf{0}$ (fixed under *Gale transform*);
2. Subsets of $2n-1$ points impose the same number of conditions on quadrics as Γ:
$$I(\Gamma \setminus \gamma)_2 = I(\Gamma)_2 \qquad \forall \gamma \in \Gamma;$$
3. $\exists Q \in \mathrm{Sym}(n)$ non-deg. and $\Gamma = \Gamma_1 \dot{\cup} \Gamma_2$ s.t. Γ_1, Γ_2 are orthogonal bases w.r.t. Q:
$$\Gamma_i^{\mathsf{T}} \cdot Q \cdot \Gamma_i \in \mathrm{Diag}(n), \qquad i = 1, 2.$$

If Γ fails to impose indep. cond. on quadrics by 1:

4. [Eisenbud & Popescu] The homogeneous coordinate ring $S_\Gamma = \mathbb{C}[\underline{x}]/I(\Gamma)$ is Gorenstein.

⇝ Slices of canonical curves are self-dual!



A parametrization of the Moduli space

Let $\mathcal{A}_{n-1} \subseteq (\mathbb{P}^{n-1})^{2n} /\!/ \operatorname{SL}_n \times \mathfrak{S}_{2n}$ be the Moduli space of self-dual points

1. All sets of four points in $\mathbb{P}^1$ are self-dual
2. Six points in $\mathbb{P}^2$ are self-dual iff intersection of quadric and cubic
3. A general set in $\mathcal{A}_3$ is a complete intersection of three quadric surfaces in $\mathbb{P}^3$
4. ...in $\mathcal{A}_4$ is a section of $X_6 = \operatorname{Gr}(2, \mathbb{C}^5) \subseteq \mathbb{P}^9$ with a quadric and a linear space
5. ...in $\mathcal{A}_5$ is a linear section of $X_7 = \operatorname{LG}_+(5, \mathbb{C}^{10}) \subseteq \mathbb{P}^{15}$
6. ...in $\mathcal{A}_6$ is a linear section of the Grassmannian $X_8 = \operatorname{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{P}^{14}$

1.–4. classical/[Eisenbud & Popescu 2000], 5.–6. [Petrakiev 2006], fails for $\mathcal{A}_7$

The Mukai lifting (and slicing) problem

$X_8 = \mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{P}^{14}$, $\operatorname{codim} X_8 = 6$, $\deg X_8 = 14$

- Slicing: Given a linear space $\mathbb{L} \subseteq \mathbb{P}^{14}$, compute the self-dual point configuration

$$\Gamma = \mathbb{L} \cap \mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{L} \cong \mathbb{P}^6$$

- Lifting: Given a self-dual points $\Gamma \subseteq \mathbb{P}^6$, find a $\mathbb{L} \in \mathbb{Gr}(6, \mathbb{P}^{14})$ and $L \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L}$

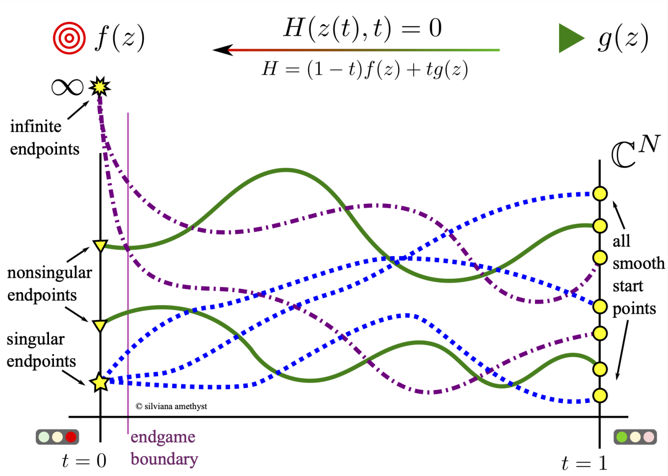
$$\Gamma = L^{-1}(\mathbb{L} \cap \mathrm{Gr}(2, \mathbb{C}^6))$$

- Numerical or symbolic? Complex or real?
- Also interesting for other X_g, or for canon. curves, K3 surfaces, Fano 3-folds, ...
- Computational problem posed by [Geiger, Hashimoto, Sturmfels & Vlad 2022]

That one slide about homotopy continuation

- ▷ Slicing: Move linear space through Grassmannian
- ⇝ track intersection points

- ▷ Lifting: Move point configuration through $\mathcal{A}_6$ (?)
- ⇝ track some linear space in fiber of "slicing map"



Credit: silviana amethyst

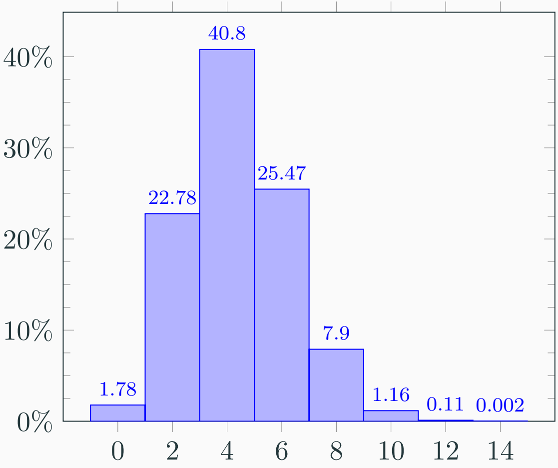
Warm-up: Slicing $X_8 = \mathrm{Gr}(2, \Bbbk^6)$

Given $\mathbb{L} = \mathrm{Ker}\, A \subseteq \mathbb{P}^{14}$, find $\mathbb{L} \cap X_8$

- Toric degeneration of $\mathrm{Gr}(2, \mathbb{C}^6)$ via SAGBI basis

1. Solve for random $\mathbb{L}_0$ on toric variety (polyhedral start system)
2. Track via toric degeneration to X_8
3. Track $\mathbb{L}_0 \to \mathbb{L}$ (straight line homotopy)

Application: Slice $\mathrm{Gr}(2, \mathbb{R}^6) \subseteq \mathbb{P}^{14}(\mathbb{R})$ with 10,000,000 $\mathbb{L} \in \mathbb{Gr}(6, \mathbb{P}^{14}(\mathbb{R}))$ sampled uniformly, count real solutions



Parametrization of $\mathcal{A}_n$

- $\mathcal{A}_n$ known to be rational variety [Dolgachev & Ortland]
- Orthogonal normal form: Non-degenerate self-dual points has representation

$$\Gamma = [I_n \mid P], \qquad P \in \mathrm{SO}(n, \mathbb{C})$$

- Cayley transform: Let $U = \{\, A \in \mathbb{C}^{n\times n} \mid I_n + A \text{ invertible} \,\}$

$$\mathcal{C}\colon U \cap \mathrm{Skew}(n) \leftrightarrow U \cap \mathrm{SO}(n), \qquad \mathcal{C}(A) = (I_n - A)(I_n + A)^{-1}$$

- Skew normal form: General self-dual points have representation by $S \in \mathrm{Skew}(n)$

$$\Gamma = [I_n + S \mid I_n - S] = \left[\begin{array}{cccc|cccc} 1 & s_1 & \cdots & s_{n-1} & 1 & -s_1 & \cdots & -s_{n-1} \\ -s_1 & 1 & \ddots & \vdots & s_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_{\binom{n}{2}} & \vdots & \ddots & \ddots & -s_{\binom{n}{2}} \\ -s_{n-1} & \cdots & -s_{\binom{n}{2}} & 1 & s_{n-1} & \cdots & s_{\binom{n}{2}} & 1 \end{array}\right]$$

- Highly non-unique (5,579,410,636,800 SNFs for $\mathbb{P}^6$), but *linear* in $S = (s_1, \ldots, s_{21})$

A big polynomial system

- ▷ Gen. finite slicing map f
  $\mathbb{L} \mapsto \mathbb{L} \cap \mathrm{Gr}(2, \mathbb{C}^6)$
- ▷ Lifts to $\hat{f}$ on matrices:
  $L \mapsto L^{-1}(\mathrm{Im}(L) \cap \mathrm{Gr}(2, \mathbb{C}^6))$
- ▷ General fiber of $\hat{f}$ 36 dim'l
- ⇝ L_a should have 69 free vars
- ▷ Polynomial system in (a, t)

$$\text{plück}_i(L_a(\mathrm{SNF}(S_t)_j)) = 0$$

$$i = 1, \ldots, 15 \text{ rel's}$$

$$j = 1, \ldots, 14 \text{ pts}$$

$$\begin{tikzcd}
 & & \widehat{\mathcal{A}}_6 \subseteq (\mathbb{P}^6)^{14} \arrow[d, two heads] & \\
\mathbb{C}^{7\times 15} \arrow[rr, dashed, two heads, "\hat{f}"] \arrow[d, dashed, two heads] & & \widehat{\mathcal{A}}_6/\mathfrak{S}_{14} \subseteq (\mathbb{P}^6)^{14}/\!/\mathfrak{S}_{14} \arrow[dd, two heads] & \mathrm{Skew}(7) \arrow[ul, dashed, two heads, "\text{SNF}"'] \arrow[ddl, dashed, two heads, "\text{gen.fin}"] \\
\mathrm{Gr}(7, \mathbb{C}^{15}) \arrow[d, dashed, two heads] & \circlearrowleft & & \\
\mathrm{Gr}(7, \mathbb{C}^{15})/\,\mathrm{SL}_6 \arrow[rr, dashed, two heads, "f", "\text{gen.fin}"'] & & \mathcal{A}_6 \subseteq (\mathbb{P}^6)^{14}/\!/\,\mathrm{SL}_7 \times \mathfrak{S}_{14} &
\end{tikzcd}$$

A buffet of mathematical software

julia

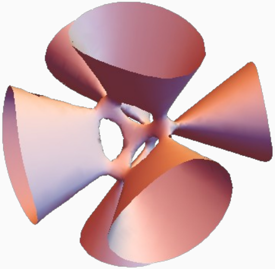
Homotopy Continuation.jl

OSCAR
SYMBOLIC TOOLS

MATHREPO
MATHEMATICAL RESEARCH-DATA REPOSITORY

What's next?

- Methods apply to "smaller" Mukai Grassmannians too
- Improve runtime!
- Test Petrakiev's birationality conjecture $\mathbb{G}\mathrm{r}(6,\mathbb{P}^{14})/\,\mathrm{SL}_6 \overset{\sim}{\dashrightarrow} \mathcal{A}_6$ (ongoing)
- Lifting real/rational solutions to real/rational linear spaces?
- Attack Mukai lifting problem for canonical curves?
  - $\rightsquigarrow$ Lifting of 0-dim'l slices could be stepping stone!



Thank you!

`arXiv:2406.02734`

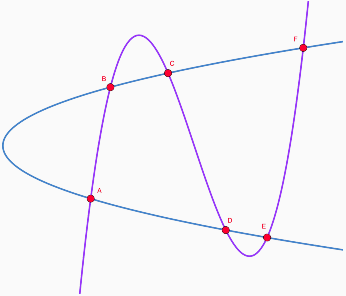


Image credit

- Slide 1: https://en.wikipedia.org/wiki/K3_surface#/media/File:K3_surface.png
- Slide 2: Made using GeoGebra https://www.geogebra.org/graphing
- Slide 5: silviana amethyst https://silviana.org/computer_programs/

- Barbara Betti and Leonie Kayser.
  Mukai lifting of self-dual points in $\mathbb{P}^6$.
  2024.
- Igor Dolgachev and David Ortland.
  Point sets in projective spaces and theta functions.
  Number 165 in Astérisque. Société mathématique de France, 1988.
- David Eisenbud and Sorin Popescu.
  The projective geometry of the Gale transform.
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Self-dual matroids from canonical curves.
Experimental Mathematics, 2022.

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Curves, K3 surfaces and Fano 3-folds of genus ≤ 10.
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On self-associated sets of points in small projective spaces.
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