Commutative and non-commutative rank

Seminar on Tensor Ranks and Tensor Invariants

Leonie Kayser
leokayser.github.io

July 18, 2024

Today's menu

3-tensor	$T = [a_{hij}] \in \mathbb{F}^a \otimes \mathbb{F}^b \otimes \mathbb{F}^c$	$\cdots \rightsquigarrow$ equations for secant varieties
	$\downarrow\uparrow$	
linear matrix	$M = [\ell_{ij}(\underline{x})] \in \mathbb{F}[\underline{x}]_1^{n \times m}$	(non)-commutative rank
	$\downarrow\uparrow$	
linear space of matrices	$\mathcal{H} = \langle M_1, \ldots, M_d \rangle_{\mathbb{F}} \subseteq \mathbb{F}^{n \times m}$	max-rank, blow-up decomposable spaces

Maximal rank

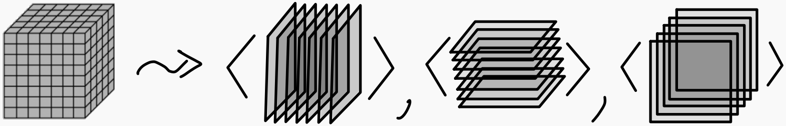
- A, B, C fin. vector spaces over field $\mathbb{F}$, $T = \sum_i a_i \otimes b_i \otimes c_i \in A \otimes B \otimes C$
- Fulvio: First standard flattening

$$T_A \colon A^* \to B \otimes C = \mathrm{Hom}(B^*, C), \qquad \alpha \mapsto \alpha(T) = \sum_i \alpha(a_i) \cdot b_i \otimes c_i$$

- Flattening rank: Matrix rank of $T_A \in \mathrm{Hom}(A^*, B \otimes C)$ $(= \dim_{\mathbb{F}} \mathrm{Im}(T_A))$

Definition (max rank aka. commutative rank)

The *max ranks* of T are $\mathrm{maxrk}_A\, T = \max_{M \in \mathrm{Im}(T_A)} \mathrm{rk}\, M$, similarly for B, C



Linear matrices and commutative rank

- Let $\mathcal{H} = \langle M_0, \ldots, M_d \rangle_{\mathbb{F}} \subseteq \mathbb{F}^{n\times m}$ be a linear space of matrices
- The *rank* of $\mathcal{H}$ is $\operatorname{maxrk} \mathcal{H} := \max_{H\in\mathcal{H}} \operatorname{rk} H$
- Consider $M := M_0 + M_1 x_1 + \cdots + M_d x_d \in \mathbb{F}[\underline{x}]^{n\times m}$

Lemma

$\operatorname{maxrk} \mathcal{H} = \operatorname{rk}_{\mathbb{F}(\underline{x})} M$ *if* $|\mathbb{F}|$ *is sufficiently large (always have "*$\leq$*").*

- Let $\mathcal{M}in_r(-)$ be the set of r-minors and $M' = \sum_{i=0}^{d} M_i x_i \in \mathbb{F}[x_0, \ldots, x_d]_1^{n\times m}$, then

$$\operatorname{rk}_{\mathbb{F}(\underline{x})} M < r \quad \Leftrightarrow \quad \mathcal{M}in_r\, M = \{0\} \quad \Leftrightarrow \quad \mathcal{M}in_r\, M' = \{0\}$$

- Minors of M' are homog. poly. of degree r, identically zero iff zero on $\mathbb{P}(\mathbb{F}^{d+1})$
- These are all minors of all elements $\mathcal{H}$, so vanish iff $\operatorname{maxrk} \mathcal{H} < r$ □

$\rightsquigarrow$ $\operatorname{maxrk}_A T = \operatorname{rk}_{\mathbb{F}(\underline{x})} M$, where M is any linear matrix obtained from $\mathcal{H} = \operatorname{Im}(T_A)$

Towards non-commutative rank

- 3-tensors give rise to linear (spaces of) matrices (later more)
- Continue notation $M = M_0 + M_1 x_1 + \cdots + M_d x_d$, $\mathcal{H} = \langle M_0, \ldots, M_d \rangle_{\mathbb{F}}$
- Embed $\langle 1, x_1, \ldots, x_d \rangle_{\mathbb{F}}$ into a field other than $\mathbb{F}(x_1, \ldots, x_d)$, and consider rank there?
- But rank of a matrix is invariant under field extensions. . .

But what if it's not a commutative field?

Interlude: Linear algebra over division rings

Let $\mathbb{D}$ be a *division ring* (ring s.t. all $x \in \mathbb{D} \setminus 0$ have multiplicative inverse)

- A $\mathbb{D}$-vector space V is a right module over $\mathbb{D}$: $\binom{x}{y} \cdot \lambda = \binom{x\lambda}{y\lambda}$
- Linearly independent subset can be extended to basis generating sets contain bases
- $\dim_{\mathbb{D}} V = |\text{basis}|$ well-defined, will only consider $< \infty$
- For $B \subseteq V$ two of $\{|B| = \dim_{\mathbb{D}} V, \langle B \rangle_{\mathbb{D}} = V, B \text{ independent}\}$ imply third
- $\dim_{\mathbb{D}} V = \dim_{\mathbb{D}} W$, then $f \colon V \to W$ injective $\Leftrightarrow$ surjective $\Leftrightarrow$ invertible
- Linear maps $\mathbb{D}^m \to \mathbb{D}^n$ represented by left-multiplication of matrix $A \in \mathbb{D}^{n \times m}$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right) \cdot \lambda \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} \cdot \lambda \right)$$

Matrix rank over division rings

Let $f\colon V \to W$ is $\operatorname{rk} f := \dim_{\mathbb{D}} \operatorname{Im}(f)$, be represented by $A \in \mathbb{D}^{n\times m}$, then

$$
\begin{aligned}
& \operatorname{rk} f := \dim_{\mathbb{D}} \operatorname{Im}(f) \\
= \;& \#\ \text{linearly independent columns (as right-vectors, } \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \cdot \lambda) \\
= \;& \#\ \text{linearly independent rows (as left-vectors, } \lambda \cdot \left(\begin{smallmatrix} x & y \end{smallmatrix}\right)) \\
= \;& \text{Minimal } r \text{ such that } A = BC,\ B \in \mathbb{D}^{n\times r},\ C \in \mathbb{D}^{r\times m} \text{ “inner rank” } \operatorname{innrk}_R A \\
= \;& \text{Maximal } r \text{ such that } PAQ = I_r,\ P \in \mathbb{D}^{r\times n},\ Q \in \mathbb{D}^{m\times r} \\
\neq \;& \operatorname{rk} A^{\mathsf{T}} \text{ in general! Reason: } f^* = A^{\mathsf{T}} \text{ a } \mathbb{D}^{\mathrm{op}}\text{-linear map, do not consider over } \mathbb{D}!
\end{aligned}
$$

Counterexample over *quaternions* $\mathbb{H} = \langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k} \rangle_{\mathbb{R}}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$

$$
\operatorname{Im} \begin{bmatrix} 1 & \mathrm{j} \\ \mathrm{i} & \mathrm{k} \end{bmatrix} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathbb{H}, \qquad \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{j} & \mathrm{k} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -\mathrm{j} \\ -\mathrm{i} & -\mathrm{k} \end{bmatrix} = I_2
$$

The free division algebra $\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle$

Theorem (Cohn)

$\mathbb{F}\langle \underline{x} \rangle := \mathbb{F}\langle x_1, \ldots, x_d \rangle$ *can be embedded into a division ring* $\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle$ *such that*

1. $\mathbb{F}\langle \underline{x} \rangle$ *generates* $\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle$ *as a division ring (smallest division subring containing …)*
2. *All matrices* $A \in \mathbb{F}\langle \underline{x} \rangle^{n \times n}$ *with* $\operatorname{innrk}_{\mathbb{F}\langle \underline{x} \rangle} A = n$ *become invertible over* $\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle$

Moreover $\operatorname{rk}_{\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle} B = \operatorname{innrk}_{\mathbb{F}\langle \underline{x} \rangle} B$ *for all* $B \in \mathbb{F}\langle \underline{x} \rangle^{n \times m}$

- Construct as subalgebra $\mathbb{F}\langle \underline{x} \rangle \subseteq \mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle \subseteq$ “non-commutative formal laurent series”
- Every $y \in \mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle$ occurs as an element of the inverse of a matrix $A \in \mathbb{F}\langle \underline{x} \rangle_{\leq 1}^{n \times n}$
- Alternative universal property using specializations

Non-commutative rank and r-decomposable linear spaces

Definition (r-decomposable)

A subspace of matrices $\mathcal{H} = \langle M_0, \ldots, M_d \rangle_{\mathbb{F}} \subseteq \mathbb{F}^{n \times m}$ is *r-decomposable*, $r \leq m, n$, if there are invertible matrices P, Q over $\mathbb{F}$ such that

$$PHQ \subseteq \begin{bmatrix} \mathbb{F}^{n-i \times j} & 0^{n-i \times m-j} \\ \mathbb{F}^{i \times j} & \mathbb{F}^{i \times m-j} \end{bmatrix}, \qquad i + j = r.$$

Theorem (Fortin & Reutenauer)

$\mathcal{H}$ is r-decomposable if and only if $\mathrm{rk}_{\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle} M \leq r$ where $M = M_0 + M_1 x_1 + \cdots + M_d x_d$.

$$\text{“}\Rightarrow\text{”}: \quad \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & I_i \end{bmatrix} \cdot \begin{bmatrix} I_j & 0 \\ 0 & C \end{bmatrix} \quad \rightsquigarrow \quad \mathrm{innrk}_{\mathbb{F}\langle \underline{x} \rangle} M \leq r. \qquad \text{“}\Leftarrow\text{”}: \quad \text{harder.}$$

Commutative vs non-commutative rank

Definition ((Non-)commutative rank)

For a linear matrix $M \in \langle 1, x_1, \ldots, x_d \rangle_{\mathbb{F}}^{n \times m}$ the (non-)commutative ranks are

$$\operatorname{crk} M = \operatorname{rk}_{\mathbb{F}(\underline{x})} M, \qquad \operatorname{ncrk} M = \operatorname{rk}_{\mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle} M$$

- Let $\mathcal{H} \subseteq \mathbb{F}^{n \times m}$ be the associated matrix subspace, then if $\mathbb{F}$ is sufficiently large

$$\operatorname{crk} M = \max \{ \operatorname{rk} H \mid H \in \mathcal{H} \}, \quad \operatorname{ncrk} M = \min \{ r \geq 0 \mid \mathcal{H} \text{ is } r\text{-decomposable} \}$$

- In particular have inequalities $0 \leq \operatorname{crk} M \leq \operatorname{ncrk} M \leq \min\{m, n\}$
- Fortin & Reutenauer: $\operatorname{crk} M = \operatorname{ncrk} M$ iff $\mathcal{H}$ is *compression space* or of full rank
- One can show: If $\operatorname{maxrk} \mathcal{H} = r$, then $\mathcal{H}$ is $2r$-decomposable with $i = j = r$, so

$$\operatorname{crk} M \leq \operatorname{ncrk} M \leq 2 \operatorname{crk} M$$

- If $M \neq 0$, then actually $1 \leq \frac{\operatorname{ncrk} M}{\operatorname{crk} M} < 2$. Can we do better?

The worst-case ratio

- If $\operatorname{crk} M = 1$, then all $\mathcal{H}$ are dependent, so $\mathcal{H}$ is 1-decomposable and $\operatorname{ncrk} M = 1$
- The following linear matrix has $\operatorname{crk} M = 2$, but $\operatorname{ncrk} M = 3$:

$$M = \begin{bmatrix} 0 & 1 & x_1 \\ -1 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix} \overset{\text{over } \mathbb{F}\langle\!\langle \underline{x} \rangle\!\rangle}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & [x_2, x_1] \end{bmatrix}, \qquad [x_2, x_1] = x_2 x_1 - x_1 x_2 \neq 0$$

- Derksen & Makam: Linear matrices $A(p,d)$ of format $\binom{d}{p} \times \binom{d}{p+1}$ such that

$$\frac{\operatorname{ncrk} A(p, 2p+1)}{\operatorname{crk} A(p, 2p+1)} = \frac{2p+1}{p+1} \xrightarrow{p \to \infty} 2$$

- Construction: $V \cong \mathbb{Q}^d$, then $A(p,d)$ corresponds to image of linear map

$$L \colon V \to \operatorname{Hom}\Big(\bigwedge^p V, \bigwedge^{p+1} V\Big), \qquad v \mapsto L_v = (w \mapsto v \wedge w)$$

- $\operatorname{crk} A(p,d) = \binom{d-1}{p}$, $\operatorname{ncrk} A(p, 2p+1) = \binom{2p+1}{p} = \frac{2p+1}{p+1}\binom{2p}{p}$

Tensor blow-ups

Definition (Tensor blow up)

Given $\mathcal{H} \subseteq \mathbb{F}^{n\times m}$ or M, the (p,q)-th *tensor blow-up* is

$$\mathcal{H}^{\{p,q\}} := \mathcal{H} \otimes \mathbb{F}^{p\times q} = \left\{ \sum_i M_i \otimes X_i \;\middle|\; X_i \in \mathbb{F}^{p\times q} \right\} \subseteq \mathbb{F}^{np\times mq}, \quad \mathcal{H}^{\{k\}} := \mathcal{H}^{\{k,k\}}.$$

- If $H \in \mathcal{H}$ has rank r, then $I_k \otimes H \in \mathcal{H}^{\{k\}}$ has rank kr
- In particular $\operatorname{maxrk} \mathcal{H}^{\{k\}} \geq k \cdot \operatorname{maxrk} \mathcal{H}$
- Example: $\mathcal{H} = \operatorname{Skew}(3) \subseteq \mathbb{F}^{3\times 3}$ with basis M_0, M_2, M_2 satisfies

$$\operatorname{maxrk} \mathcal{H} = 2, \qquad \operatorname{rk}\left(M_0 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + M_1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + M_2 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = 6 > 4$$

Another characterization of non-commutative rank

- Regularity lemma: $\operatorname{maxrk} \mathcal{H}^{\{k\}}$ is always a multiple of k
- For $k > \frac{\min\{m,n\}}{2}$ the sequence $a_k = \frac{\operatorname{maxrk} \mathcal{H}^{\{k\}}}{k}$ is weakly increasing
- In fact, $(a_k)_k$ constant for $k > \min\{m, n\}$

Theorem

$$\operatorname{ncrk} M = \lim_{k\to\infty} \frac{\operatorname{maxrk} \mathcal{H}^{\{k\}}}{k} = \max_{k\geq 1} \frac{\operatorname{maxrk} \mathcal{H}^{\{k\}}}{k}$$

- Idea: Let $T_1, \ldots, T_d \in \mathbb{F}[\{ t_{ij}^h \mid 1 \leq i, j \leq k,\ h \in \mathbb{N} \}]^{k\times k}$ be generic matrices, then

$$\operatorname{maxrk} \mathcal{H}^{\{k\}} = \operatorname{rk}_{\mathbb{F}(\{t_{ij}^h\})} \left(M_0 \otimes I_k + M_1 \otimes T_1 + \cdots + M_d \otimes T_d\right)$$

- Approximate non-commutativity of $\mathbb{F}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$ by tensor blow-ups □

A flattening déja-vu

$T = \sum_i s_i \otimes M_i \in A \otimes (B \otimes C)$

- Identify $B \otimes C \cong \mathbb{F}^{b \otimes c}$, let $L \colon A \to \mathbb{F}^{p \otimes q}$ be a linear map, $\mathcal{H}_L := \operatorname{Im} L$
- Consider linear map

$$\psi_L \colon A \otimes B \otimes C \to \mathbb{F}^{pb \times qc}, \qquad \sum_i s_i \otimes M_i \mapsto \sum_i L(s_i) \otimes M_i$$

Lemma

$$\operatorname{rk}(\psi_L(T)) \leq \overline{\mathrm{R}}(T) \cdot \operatorname{maxrk} \mathcal{H}_L$$

- If $T = s_1 \otimes (a_1 \otimes b_1)$, then $\psi_L(T) = L(s_1) \otimes (a_1 \otimes b_1)$, hence $\operatorname{rk} \psi_L(T) \leq \operatorname{rk} L(s_1) \leq \operatorname{maxrk} \mathcal{H}_L$
- If T has rank r, then by linearity $\operatorname{rk} \psi_L(T) \leq r \operatorname{maxrk} \mathcal{H}_L$
- By closedness of matrix rank this still holds for border rank

$\square$

Equations for secant varieties

$$\mathcal{H}_L = \mathrm{Im}(L\colon A \to \mathbb{F}^{p\times q}), \qquad \psi_L\colon A\otimes B\otimes C \to \mathbb{F}^{pb\times qc}, \quad \sum_i s_i \otimes M_i \mapsto \sum_i L(s_i)\otimes M_i$$

Lemma

Let $D = r\,\mathrm{maxrk}\,\mathcal{H}_L$.

1. *The* $(D+1)$*-minors of* $\psi_L(T)$ *give equations vanishing on* $\sigma_r(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C))$.
2. *One of the* $(D+1)$*-minors of* $\psi_L(T)$ *is non-trivial if and only if* $\mathrm{maxrk}\,\mathcal{H}_L^{\{b,c\}} > D$.

- The minors vanish on tensors of border rank by previous lemma
- $\psi_L(T) \in \mathcal{H}_L^{\{b,c\}}$, in fact $\mathrm{Im}\,\psi_L = \mathcal{H}_L^{\{b,c\}}$
- Assume $\mathrm{maxrk}\,\mathcal{H}_L^{\{b,c\}} > D$, then there is a $T_1 \in A\otimes B\otimes C$ with $\mathrm{rk}\,\psi_L(T_1) > D$
- Hence $\mathcal{M}in_{D+1}\,\psi_L(T_1) \neq 0$; converse follows since $\mathbb{F}$ is sufficiently large $\square$

Non-commutative rank ratio yields equations?! $A = B = C = \mathbb{F}^m$

Apply method to $L\colon \mathbb{F}^m \to \mathrm{Hom}(\bigwedge^p \mathbb{F}^m, \bigwedge^{p+1} \mathbb{F}^m) \cong \mathbb{F}^{\binom{m}{p} \times \binom{m}{p+1}}$, $m = 2p+1$

Theorem

Let $D = \binom{2p}{p}(2m-4)$, then at least one $(D+1)$-minor gives a non-trivial equation for $\sigma_{2m-4}(\mathbb{P}(\mathbb{F}^m) \times \mathbb{P}(\mathbb{F}^m) \times \mathbb{P}(\mathbb{F}^m))$.

- $\operatorname{maxrk} \mathcal{H}_L = \binom{2p}{p}$, $r = 2m - 4$ ✓
- Remains to show that $\operatorname{maxrk} \mathcal{H}_L^{\{m\}} > D$
- Let $r(p,q) = \operatorname{maxrk} \mathcal{H}_L^{\{p,q\}}$, then r is increasing and concave in each variable
- Chasing inequalities from $r(p+1, p+1)$ and $r(2p+2, 2p+2)$ eventually yields bound on $r(2p+1, 2p+1) = \operatorname{maxrk} \mathcal{H}_L^{\{m\}}$ □

More applications & thoughts

- Non-commutative rank less than twice the commutative rank, so

$$\frac{\operatorname{maxrk} \mathcal{H}_L^{\{m\}}}{\operatorname{maxrk} \mathcal{H}_L} \leq \frac{m \operatorname{maxrk} \mathcal{H}_L}{\operatorname{maxrk} \mathcal{H}_L} < 2m$$

- Can not go beyond $2m-1$ with this method, experimental evidence suggests it may work for $2m-2$
- Using this method one can explicitly construct tensors of border rank $\geq 2m-3$
- Landsberg gave equations for σ_{2m-3} and tensors of border rank $\geq 2m-2$ (m even)
- Major open problem: Find explicit tensors of super-linear border rank

Thank you!

3-tensor	$T = [a_{hij}] \in \mathbb{F}^a \otimes \mathbb{F}^b \otimes \mathbb{F}^c$	$\cdots \rightsquigarrow$ equations for secant varieties
	$\downarrow\uparrow$	
linear matrix	$M = [\ell_{ij}(\underline{x})] \in \mathbb{F}[\underline{x}]_1^{n \times m}$	(non)-commutative rank
	$\downarrow\uparrow$	
linear space of matrices	$\mathcal{H} = \langle M_1, \dots, M_d \rangle_{\mathbb{F}} \subseteq \mathbb{F}^{n \times m}$	max-rank, blow-up decomposable spaces

References i

P.M. Cohn.
Free Rings and Their Relations.
LMS monographs. Academic Press, 1971.

Harm Derksen and Visu Makam.
On non-commutative rank and tensor rank.
Linear and Multilinear Algebra, 66:1069 – 1084, 2016.

Marc Fortin and Christophe Reutenauer.
Commutative/noncommutative rank of linear matrices and subspaces of matrices of low rank.
Séminaire Lotharingien de Combinatoire [electronic only], 52:B52f, 12 p., electronic only–B52f, 12 p., electronic only, 2004.

Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson.
Operator scaling: Theory and applications.
Foundations of Computational Mathematics, 20(2):223–290, May 2019.

Gábor Ivanyos, Youming Qiao, and K. V. Subrahmanyam.
Non-commutative edmonds' problem and matrix semi-invariants.
computational complexity, 26(3):717–763, August 2016.