Recovering forms from their space of partial derivatives

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Euler's formula and wishful thinking

Given the partial derivatives of a homogeneous polynomial $F \in \mathbb{C}[x_0,\dots,x_n]_D$, one can use integration to recover the potential F. In fact, one of the many Euler formulas gives a way to express F as a combination of its partials:

$$F = \frac{1}{D}\sum_{i=0}^{n} x_i \cdot \frac{\partial}{\partial x_i}F.$$

💡 *How does a higher order partials version of Euler's formula look like?*

However, if only the vector space of *all* directional derivatives

$$\langle \nabla^1 F\rangle := \{\, \partial_v F \mid v \in \mathbb{C}^{n+1} \,\} = \langle \tfrac{\partial}{\partial x_0}F,\dots,\tfrac{\partial}{\partial x_n}F\rangle_{\mathbb{C}}$$

is known, then recovering F is generally no longer possible, not even up to scaling!

Example. For any $a,b,c \in \mathbb{C}^\times$ we have

$$F = ax^3 + by^3 + cz^3 \quad\rightsquigarrow\quad \langle \nabla^1 F\rangle = \langle x^2, y^2, z^2\rangle_{\mathbb{C}}.$$

On the other hand, $G = xyz$ is the only form with $\langle \nabla^1 G\rangle = \langle xy, xz, yz\rangle_{\mathbb{C}}$. 💡 *Why?*

This leads to some interesting questions:

1. Can a general form F be recovered from its spaces of higher order partials $\langle \nabla^d F\rangle$?
2. If not, how does the set of polynomials with the same partials look like?
3. And which subspaces $\Gamma \subseteq \mathbb{C}[x_0,\dots,x_n]_{D-d}$ are spaces of partials?

Apolarity: The differentiation action

Consider a finite $\mathbb{C}$-vector space V, let

- $T := \mathrm{S}(V) = \bigoplus_D \mathrm{S}^D(V)$ be the symmetric tensor algebra and
- $S := \mathrm{Sym}(V^*) = \bigoplus_d \mathrm{Sym}^d(V^*)$ the symmetric algebra.

After choosing a basis $V = \langle x_0,\dots,x_n\rangle_{\mathbb{C}}$, we can identify $T = \mathbb{C}[x_0,\dots,x_n]$ and $S = \mathbb{C}[\partial_0,\dots,\partial_n]$, where $\partial_i = x_i^*$. Tensor contraction defines the apolar pairing $S_d \times T_D \to T_{D-d}$, which under this identification corresponds to differentiation

$$\partial^\alpha \bullet x^\beta = \frac{\beta!}{(\beta-\alpha)!}x^{\beta-\alpha} \text{ if } \alpha \le \beta, \text{ else } 0.$$

For $d = D$ this is a perfect pairing, identifying $(\mathrm{S}^d V)^* = \mathrm{Sym}^d(V^*)$, and in general defines a S-module structure on T (!). In this sense S is the commutative ring of partial differential operators with constant coefficients on T.

💡 *Is T finitely generated over S? Dare to think about positive characteristics?*

The *annihilator* of $F \in T_D$, i. e. all operators which kill F, is the apolar ideal of F

$$F^\perp := \mathrm{Ann}_S(F) = \{\, g \in S \mid g \bullet F = 0 \,\}.$$

The graded ideal $F^\perp$ equals S in degrees $> D$, and the quotient $S/F^\perp$ is a graded Artinian Gorenstein $\mathbb{C}$-algebra of socle degree D (see for example [1]).

Theorem. *(Macaulay) The map $F \mapsto S/F^\perp$ induces a bijection*

$$\mathbb{P}(T_D) \to \{\textit{Graded Gorenstein Artinian quotients of } S \textit{ of socle deg. } D\}.$$

💡 *Can you show it's injective? Hint: Look at the degree D component.*

Forms with spaces of partials of given dimension

For fixed $F \in T_D$ and any $0 < d < D$, the apolar paring defines the linear Catalecticant map sending a form $g(\partial)$ to the derivative of F with respect to g

$$F_{d,D-d}\colon S_d \to T_{D-d}, \qquad g \mapsto g \bullet F.$$

The image of $F_{d,D-d}$ is exactly its space of d-th order partials, while the kernel is the d-th graded component of $F^\perp$ (by definition). In particular, we have a short exact sequence

$$0 \longrightarrow (F^\perp)_d \hookrightarrow S_d \xrightarrow{F_{d,D-d}} \langle \nabla^d F\rangle \longrightarrow 0.$$

Thus image, kernel and rank of $F_{d,D-d}$ are of great interest for our question. Varying F, we would like to consider a morphism mapping $F \mapsto [\mathrm{Im}\, F_{d,D-d}]$ into some Grassmannian. To make this idea rigorous, we need to introduce the appropriate domains of definition

$$\mathrm{Sub}^\circ_{d,k} := \{\, [F] \in \mathbb{P}(T_D) \mid \mathrm{rk}\, F_{d,D-d} = k \,\},$$
$$\mathrm{Sub}_{d,k} := \overline{\mathrm{Sub}^\circ_{d,k}} = \{\, [F] \in \mathbb{P}(T_D) \mid \mathrm{rk}\, F_{d,D-d} \le k \,\}$$

for $1 \le k \le k_{\max} := \min\{\binom{n+d}{n}, \binom{n+D-d}{n}\}$.

💡 *Why this $k_{\max}$? Can $\mathrm{Sub}^\circ_{d,k}$ be empty?*

Example. For a form $F \in T_D$ let $\mathrm{Supp}\, F$ be the smallest subspace $U \subseteq V = \mathbb{C}^{n+1}$ such that $F \in \mathrm{S}^D U$, so F is a polynomial on the (smaller) subspace U. Then

$$\mathrm{Sub}_{1,k} = \{\, [F] \mid \dim_{\mathbb{C}} \mathrm{Supp}\, F \le k \,\}$$

is the set of polynomials which (after a coordinate change) depend only on k variables.

Warning: For arbitrary (n,D,d,k) the varieties $\mathrm{Sub}_{d,k}$ may *not* be irreducible!

Spaces of partials as a morphism

For fixed $0 < d < D$ and $1 \le k \le k_{\max}$ consider the rational morphisms

$$\alpha_{d,k}\colon \mathrm{Sub}_{d,k} \dashrightarrow \mathrm{Gr}(k, T_{D-d}), \qquad F \mapsto [\mathrm{Im}(F_{d,D-d})]$$
$$\beta_{d,k}\colon \mathrm{Sub}_{d,k} \dashrightarrow \mathrm{Gr}\left(\tbinom{n+d}{n} - k, S_d\right), \qquad F \mapsto [\mathrm{Ker}(F_{d,D-d})]$$

with domain of definition $\mathrm{Sub}^\circ_{d,k}$. The first question on the left is *exactly*, whether $\alpha_{d,k_{\max}}$ is generically injective, while the second and third question ask about fibers and images of the rational maps $\alpha_{d,k}$.

The maps $\alpha_{d,k}$ and $\beta_{d,k}$ are closely related: The duality $\mathrm{Gr}(k,W) \cong \mathrm{Gr}(\dim_{\mathbb{C}} W - k, W^*)$ for $W = T_d$ induces the following commuting quadrangle

$$\begin{array}{ccc}
\mathrm{Sub}^\circ_{D-d,k} & \xrightarrow{\alpha_{D-d,k}} & \mathrm{Gr}(k, T_d) \\
\Big\| & & \Big\downarrow {\scriptstyle (-)^\perp}\ {\scriptstyle \cong} \\
\mathrm{Sub}^\circ_{d,k} & \xrightarrow{\beta_{d,k}} & \mathrm{Gr}\left(\tbinom{n+d}{n} - k, S_d\right)
\end{array}$$

💡 *Convince yourself that the two $\mathrm{Sub}^\circ_{d,k}$'s on the left indeed coincide.*

Thus any property of $\alpha_{d,k}$ immediately translates into a property of $\beta_{D-d,k}$ and vice versa, making it possible to use either of these two whenever convenient. The fibers of these morphism are very well-behaved:

Theorem. *For every $F \in \mathrm{Sub}^\circ_{d,k}$ the scheme-theoretic fiber $\alpha^{-1}_{d,k}(\alpha_{d,k}(F))$ is reduced and the intersection of a linear space $L \subseteq \mathbb{P}(T_D)$ with $\mathrm{Sub}^\circ_{d,k}$. In other words, the closures of the fibers are linear spaces.*

Corollary. The following are equivalent:

- $\alpha_{d,k}$ is birational onto its image;
- Each irreducible component of $\mathrm{Sub}^\circ_{d,k}$ has a F with $\alpha^{-1}_{d,k}(\alpha_{d,k}(F)) = \{F\}$ as sets.

First main result: Fibers of $\alpha_{1,k}$ for $d = 1$

Theorem. *The morphisms $\alpha_{1,k}$ are birational onto their image except in the following cases:*

1. *$D = 2$ and $2 \le k \le n+1$, then the fibers are $\alpha^{-1}_{1,k}([U]) = \mathbb{P}(\mathrm{S}^2 U) \subseteq \mathbb{P}(\mathrm{S}^2 V)$.*
2. *$D = 3$ and $k = 2$, then fibers are one-dimensional. Considered as points in $\mathrm{F}_1(\mathrm{Sub}_{1,2})$, they are the secant lines $\mathrm{Sec}_2 V^{n,3}$.*

Here $\mathrm{F}_1(X) \subseteq \mathbb{G}\mathrm{r}(1,\mathbb{P}(T_3))$ is the Fano scheme of lines and $\mathrm{Sec}_2 V^{n,3} \subseteq \mathbb{G}\mathrm{r}(1,\mathbb{P}(T_3))$ is the (abstract) secant variety of the Veronese variety $V^{n,3} = \nu_3(\mathbb{P}^n)$.

For $D = 3$ this was worked out previously in [2], this project is a continuation of these ideas. The key player in the proof is the form

$$F = \sum_{i=0}^{k-1} x_i^{D-1} x_{(i+1) \bmod k} \in T_D,$$

we show that (in most cases) it is the unique form with

$$\langle \nabla^1 F\rangle = \langle (D-1)x_i^{D-2}x_{i+1} + x_{i-1}^{D-1} \mid 0 \le i < k\rangle_{\mathbb{C}}$$

(indices again mod k). The variety $\mathrm{Sub}_{1,k}$ is irreducible, so the existence of F implies birationality.

💡 *Explain 1. using diagonalization of quadratic forms over $\mathbb{C}$!*

The next steps

Once the fibers and birationality are understood, the next question is studying the image of $\alpha_{d,k}$, i. e. the variety of subspaces of partials of polynomials

$$\mathcal{Z}^\circ_{d,k} := \alpha_{d,k}(\mathrm{Sub}^\circ_{d,k}), \qquad \mathcal{Z}_{d,k} := \overline{\mathcal{Z}^\circ_{d,k}}, \qquad \partial\mathcal{Z}_{d,k} := \overline{\mathcal{Z}_{d,k} \setminus \mathcal{Z}^\circ_{d,k}}$$

The main result for $d = 1$ allows us in that case to compute the dimension (in the non-exceptional cases) as

$$\dim \mathcal{Z}_{d,k} = \dim \mathrm{Sub}_{d,k} = k(n+1-k) + \binom{d+k-1}{d} - 1.$$

It would be interesting to determine the class of $\mathcal{Z}_{d,k}$ in the Chow ring of the Grassmannian. To study the boundary we introduce the Catalecticant enveloping varieties

$$\Phi_{d,k} := \{\, [\Gamma] \in \mathrm{Gr}(k, T_{D-d}) \mid \mathrm{Im}(F_{d,D-d}) \subseteq \Gamma \text{ for some } [F] \in \mathrm{Sub}_{d,k} \,\} \supseteq \mathcal{Z}_{d,k}.$$

The irreducible components $\Phi_{d,k}$ provide insight into the boundary $\partial\mathcal{Z}_{d,k}$. In a different direction, we can also study the case $n+1 = 2$ (but d arbitrary), which is closely related to Hankel matrices and Waring decomposition of binary forms.

References

[1] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, 1999.

[2] J. Sendra-Arranz, "The hessian correspondence of hypersurfaces of degree 3 and 4," 2023.