Symmetric Tensor Decomposition in small characteristics

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

Leonie Kayser

On this poster, let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geq 0$. Let V be a $\mathbb{F}$-vector space with basis $X_1, \dots, X_n$ and dual basis $x_1, \dots, x_n \in V^\vee$.

Motivation

A symmetric tensor $F \in \operatorname{Sym}^d V \subseteq V^{\otimes d}$ is a tensor invariant under the action of $\mathfrak{S}_d$. The symmetric tensor algebra $\operatorname{Sym}^* V = \bigoplus_{d \geq 0} \operatorname{Sym}^d V$ is the (graded) Hopf algebra dual of the symmetric algebra $S^* V^\vee \cong \mathbb{F}[x_1, \dots, x_n]$, and as such has the structure of an algebra. Moreover, it comes with a *divided power structure*, given by $v^{[d]} = v^{\otimes d}$ satisfying certain axioms. Every symmetric tensor $F \in \operatorname{Sym}^d V$ has a decomposition

$$F = \lambda_1 v_1^{[d]} + \dots + \lambda_r v_r^{[d]}, \qquad v_i \in V,\ \lambda_i \in \mathbb{F},$$

and the smallest possible r is the *symmetric tensor rank* $R(F)$. The smallest r such that F is in the Zariski closure of tensors of rank $\leq r$ is the *symmetric border rank* $\underline{R}(F) \leq R(F)$. This corresponds to secants of the Veronese variety $\nu_d(\mathbb{P}V) \subseteq \mathbb{P}(\operatorname{Sym}^d V)$.

If the characteristic p of $\mathbb{F}$ is 0, or at least $p > d$, then the symmetric tensor algebra is isomorphic to the symmetric algebra $\operatorname{Sym}^* V \cong S^* V$ as $\mathrm{GL}(V)$-modules and divided power algebras, the isomorphism given by

$$S^d V \ni v_1 \cdots v_d \mapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \cdots v_{\sigma(d)} \in \operatorname{Sym}^d V, \qquad \frac{1}{d!} v^d \mapsto v^{[d]}.$$

The symmetric algebra, also known as a polynomial ring, is a familiar and well-behaved object, and so most authors restrict themselves to the case of characteristic 0 or "large enough so that all troubles go away". This allows for the use of tools such as generic smoothness (in the guise of Bertini, Terracini, ...) and classical representation theory. **But what *really* happens for small p?**

Example: The symplectic block in characteristic 2

Let $\mathbb{F}$ be of characteristic 2, for example $\mathbb{F}_2^{\mathrm{alg}}$, and consider the antidiagonal symmetric matrix

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X_1 \otimes X_2 + X_2 \otimes X_1 = X^{[1,1]} \in \operatorname{Sym}^2 \mathbb{F}^2$$

It turns out that $\underline{R}(J) = 2$ and $R(J) = 3$, a minimal decomposition given by

$$J = X_1^{\otimes 2} + X_2^{\otimes 2} + (X_1 + X_2)^{\otimes 2}.$$

This funny example can be explained in at least two contexts:

- Symmetric matrices and $\mathrm{GL}(\mathbb{F}^n)$ acting on them.
- Binary (DP) forms and apolarity theory.

The case of symmetric matrices

Over a field of characteristic $p \neq 2$, any symmetric matrix A is congruent to a diagonal matrix, the number of non-zero entries being its (matrix) rank. Furthermore, the condition of having rank $\leq r$ is given by the vanishing of the $(r+1)$-minors, a closed condition. This shows

$$R(A) = \underline{R}(A) = \operatorname{rank} A.$$

Over fields of characteristic 2 the world is more interesting:

Theorem. *If $p = 2$ one has $\underline{R}(A) = \operatorname{rank} A$. The following are equivalent:*

1. $R(A) = \operatorname{rank} A$;
2. *A is diagonalizeable;*
3. *$x \mapsto x^\top A x$ is not the zero map, i.e. A has a nonzero diagonal entry*

If these equivalent conditions are not satisfied, then $R(A) = \operatorname{rank} A + 1$.

This behaviour can be explained by the group $\mathrm{GL}(\mathbb{F}^n)$ acting on symmetric matrices by congruence. A complete list for representatives of orbits on $\operatorname{Sym}^2 \mathbb{F}^n$ is given by

1. Diagonalizeable forms: $D_r = \operatorname{diag}(\underbrace{1, \dots, 1}_{r \text{ times}}, 0, \dots, 0), \quad 0 \leq r \leq n$
2. Vanishing quadratic forms: $V_{2k} = \underbrace{J \oplus \dots \oplus J}_{k \text{ times}} \oplus \mathbf{0}_{n-2k}, \quad 0 < 2k \leq n$

$$R(D_r) = \underline{R}(D_r) = r, \qquad \underline{R}(V_{2k}) = R(V_{2k}) - 1 = 2k.$$

Counterexamples to conjectures in characteristic 2

The previous result implies that many popular theorems and conjectures in characteristic 0 turn out to be false in characteristic 2 for stupid reasons:

- [Landsberg–Teitler]: Let $X \subseteq \mathbb{P}^N$ be a non-degenerate irreducible variety of dimension n. Then for all $p \in \mathbb{P}^N$, $R_X(p) \leq N + 1 - n$.
  $\lightning$ $X = \nu_2(\mathbb{P}^1) \subseteq \mathbb{P}^2 = \mathbb{P}(\operatorname{Sym}^2 \mathbb{F}^2)$, $p = [J]$, $R(J) = 3 > 2 + 1 - 1$.
- Comon Conjecture: Rank and symmetric rank of symmetric tensors agree.
  $\lightning$ $R(J) = 3 > 2 = \operatorname{rank} J$.
- Symmetric Strassen Conjecture: Symmetric tensor rank is additive.
  $\lightning$ $R(J \oplus \mathrm{Id}_1) = 3 < 4 = R(J) + R(\mathrm{Id}_1)$.

The apolar pairing

Let $S := S^* V^\vee$ be the dual polynomial ring, S acts on symmetric tensors by contraction. Consider the basis of *divided monomials* given by sums of all tensor products of the basis with prescribed "multiplicity" (but without repetition!)

$$X^{[u]} = \sum_{T \in \mathfrak{S}_d \cdot (X_1^{\otimes \alpha_1} \otimes \dots \otimes X_n^{\otimes \alpha_n})} T, \qquad \alpha \in \mathbb{N}^n,\ |\alpha| = d.$$

The DP monomials form a convenient basis of $\operatorname{Sym}^d V$, and the action of S is given by

$$x^\alpha \bullet X^{[\beta]} = X^{[\beta - \alpha]}$$

if $\alpha \leq \beta$ component-wise, and 0 otherwise. In this way, $S = S^* V^\vee$ is the canonical homogeneous coordinate ring of $\mathbb{P}(V)$. The *apolar ideal* of F is

$$F^\perp = \operatorname{Ann}_S(F) = \{\, f \in S \mid f \bullet F = 0 \,\}$$

and the *apolar algebra* is $S/F^\perp$. The relevance of this construction is illustrated by the following theorems valid over an arbitrary field (!).

Theorem. (Macaulay) *The map $\mathbb{P}(\operatorname{Sym}^d V) \ni [F] \mapsto S/F^\perp$ is a bijection onto standard graded Artin Gorenstein quotient algebras of S of socle degree d.*

Theorem. The apolarity method: *Given $0 \neq F \in \operatorname{Sym}^d V$ and $v_1, \dots, v_r \in V \setminus 0$, then the following are equivalent:*

1. $F = \lambda_1 v_1^{[d]} + \dots + \lambda_r v_r^{[d]}$ *for suitable* $\lambda_i \in \mathbb{F}$;
2. $I(\{[v_1], \dots, [v_r]\}) \subseteq F^\perp \subseteq S$;
3. $I(\{[v_1], \dots, [v_r]\})_d \subseteq F_d^\perp \subseteq S_d$.

This result brings in commutative algebra techniques, such as classifications of Gorenstein ideals of small codimension, to study symmetric tensor rank.

The case of binary DP forms

Let $F \in \operatorname{Sym}^d V$ with $\dim_{\mathbb{F}} V = 2$. By Serre's theorem, the apolar ideal is a *complete intersection* of two forms $F^\perp = \langle f, g \rangle$ of degrees $r := \deg f \leq \deg g$ with

$$\deg f + \deg g = d + 2.$$

Theorem. *Over a field of characteristic $p \geq 0$ one has*

1. $\underline{R}(F) = \deg f = \min \{\, k \mid F_k^\perp \neq 0 \,\}$
2. *If $f, g \in (\mathbb{F}[x_1, x_2]_{r/p})^p$, then $R(F) = \underline{R}(F) + 1$*
3. *Otherwise $R(F) = \deg g = d + 2 - \underline{R}(F)$*

The tensor J has apolar ideal $\langle x_1^2, x_2^2 \rangle_S$, so it has rank $d + 2 - r + 1 = 3$.

A Frobenius map on symmetric tensors

Given a symmetric tensor $F \in \operatorname{Sym}^d V$, its apolar ideal $F^\perp \subseteq S$ defines an Artin Gorenstein quotient of socle degree d. We can apply the Frobenius map $\mathrm{frob}\colon S \to S,\ f \mapsto f^p$ to this ideal to obtain another Artin Gorenstein ideal. This operation is equivalent to replacing x_i by x_i^p in equations for $F^\perp$, or by considering

$$S/F^\perp \otimes_S S^p$$

as a $S \cong S^p$-algebra.

Theorem. *By applying the Macaulay corresponence, we obtain a tensor*

$$\mathrm{frob}([F]) \in \mathbb{P}(\operatorname{Sym}^{d'} V), \qquad d' = p(d+n) - n.$$

Example. $F = X^{[1,2,4]}$ has a monomial apolar ideal

$$F^\perp = \langle x_1^2, x_2^3, x_3^5 \rangle_S \quad \rightsquigarrow \quad (F^\perp)^p = \langle x_1^{2p}, x_2^{3p}, x_3^{5p} \rangle.$$

This is the apolar ideal of the DP monomial $X^{[2p-1, 3p-1, 5p-1]}$ of degree $10p - 3$.

Observation. The exceptional cases in the previous theorem about binary DP forms are in the image of frob in $\operatorname{Sym}^d V$!

Questions

- If $p \geq 5$, then there are normal forms of tensors $F \in \operatorname{Sym}^3 \mathbb{F}^3$ up to $\mathrm{GL}(\mathbb{F}^3)$ due to the identification with $\mathbb{F}[X_1, X_2, X_3]_3$ and the theory of plane cubics. If $p \in \{2, 3\}$, then ternary cubic DP forms are not equivalent to plane cubics. Is there still a nice classification, including their ranks and border ranks?
- What is the effect of frob on symmetric tensor rank? What about iterating?
- If $p > d$, then $\operatorname{Sym}^d V \cong S^d V$ is an irreducible representation of $\mathrm{GL}(V)$, this is no longer true for smaller d. Is there a neat classification of invariant subspaces?
- The Alexander–Hirschowitz theorem shows that a general set of r double points in $\mathbb{P}(V)$ imposes independent conditions on $S^d V^*$ except in a small list of exceptions. Using apolarity one can relate this to the tangent spaces of the r-th secant variety of the Veronese varieties $\nu_d(\mathbb{P}(V)) \subseteq \mathbb{P}(\operatorname{Sym}^d V)$. This implies non-defectivity of $\sigma_r \nu_d(\mathbb{P}V)$ when $p \nmid d$. Is this result still true if $p \mid d$?
- Can we use rings of mixed characteristic like $\mathbb{Q}_p$ to prove results in characteristic 0 by lifting from characteristic $p > 0$ and vice versa?

leokayser.github.io

kayser@mis.mpg.de