Tensor Decomposition Using Numerical (Non)Linear Algebra

SIAM Conference on Applied Linear Algebra (LA24)

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

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May 14, 2024

Mathematics > Commutative Algebra

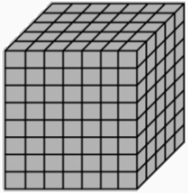
[Submitted on 5 Jul 2023]

Hilbert Functions of Chopped Ideals

Fulvio Gesmundo, Leonie Kayser, Simon Telen

A chopped ideal is obtained from a homogeneous ideal by considering
cases in which the chopped ideal defines the same finite set of points a
computing these points from the chopped ideal is governed by the Hilb
these invariants and prove them in many cases. We show that our conje
decomposition.

What is a tensor?



A tensor...

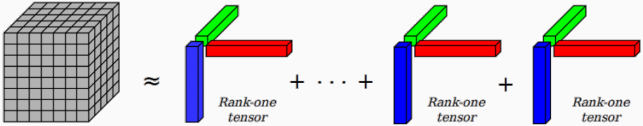
- ...is an object that transforms like a tensor
- ...is an element of a tensor product of vector spaces $U \otimes V \otimes W$
- ...is a **multidimensional array of numbers** $A = (A_{i_1 \dots i_d})_{i_1,\dots,i_d} \in \mathbb{C}^{n_1 \times \cdots \times n_d}$
- ...in $(\mathbb{C}^n)^{\otimes d}$ is symmetric if its entries are invariant under permutations $\sigma \in \mathfrak{S}_d$
- Symmetric tensors can be identified with homogeneous polynomials

$$\mathbb{C}[x_1, \dots, x_n]_d \ni x_{i_1} \cdots x_{i_d} \quad \longleftrightarrow \quad \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(d)}} \in \mathrm{Sym}^d \mathbb{C}^n \subseteq (\mathbb{C}^n)^{\otimes d}$$

Tensor decomposition and rank

- A tensor of the form $(u_i v_j w_k)_{i,j,k} \mathrel{\hat{=}} u \otimes v \otimes w$ is simple
- Every tensor is a linear combination of simple tensors

$$A = \sum_{i=1}^{r} \lambda_i u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$



- The smallest such r is the tensor rank of A
- Generalizes matrix rank: $\mathbb{C}^{m\times n} \ni A = S \cdot \mathrm{diag}(\underbrace{1,\ldots,1}_{\text{rank}\, A}, 0, \ldots) \cdot T = \sum_{i=1}^{r} S_{*,i} \cdot T_{i,*}$
- If the simple tensors are unique up to scaling, then A is called identifiable
- Symmetric case: Simple tensor $v^{\otimes d} \mathrel{\hat{=}} \ell^d$ powers of linear forms, $F = \sum_{i=1}^{r} \lambda_i \ell_i^d$
- Symmetric tensor rank, identifiability, ...

Forms of small rank often have unique decompositions

Let $T_d = \mathbb{C}[X_0, \dots, X_n]_d \cong \mathbb{C}^{\binom{n+d}{n}}$ be the vector space of degree d forms

- **(Alexander–Hirschowitz)**
  A general form $F \in T_d$ has rank $\left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$ except in a few cases
- **(Ballico, Mella, Chiantini–Ottaviani–Vannieuwenhoven, . . .)**
  For $r < \frac{1}{n+1}\binom{n+d}{d}$ a general form of rank r is identifiable except in a few cases

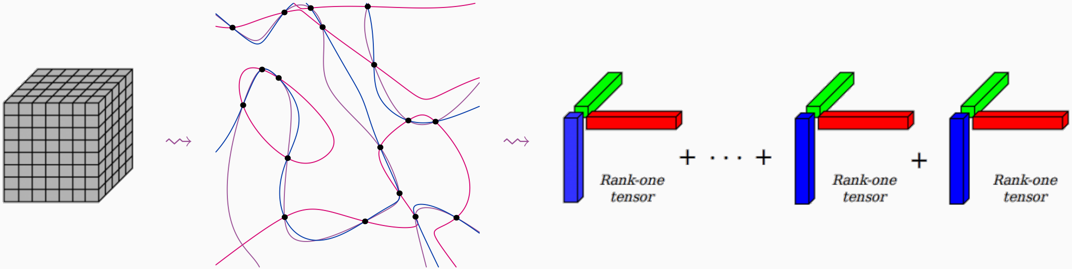
Running example

A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ has $\operatorname{rk} F = \frac{1}{3}\binom{2+10}{2} = 22$. The set of such forms of rank 18 has dimension 54 in $\mathbb{C}^{66}$. A random such F has a *unique* decomposition

$$F = \ell_1^{10} + \cdots + \ell_{18}^{10}, \qquad \ell_i \in \mathbb{C}[X_0, X_1, X_2]_1.$$

The catalecticant method

- ▷ Fix general $F = \sum_{i=1}^{r} \ell_i^d \in T_d$ of rank r
- ▷ Linear forms as points in projective space $[\ell_i] \in \mathbb{P}(T_1) \cong \mathbb{P}^n_{\mathbb{C}}$ $\qquad \mathbb{P}(V) = (V \setminus 0)/\mathbb{C}^\times$
- ▷ Catalecticant method give polynomials vanishing on $Z = \{[\ell_1], \ldots, [\ell_r]\} \subseteq \mathbb{P}^n$



- ▷ In fact: Obtain *all* homog. equations of degree $\leq d/2$ vanishing on Z

$\rightsquigarrow$ Hope: Solutions to equations are exactly the $[\ell_i]$!

The algorithm

- Equations via kernel of catalecticant maps

$$\mathrm{Cat}_j(F)\colon \mathbb{C}[y_0,\ldots,y_n]_j \to T_{d-j}, \qquad g \mapsto g(\partial_{X_0},\ldots,\partial_{X_n})F(X_0,\ldots,X_n)$$

- Algorithmic approach:
  1. Compute kernel basis $\mathcal{F}$ of the *linear* catalecticant map $\mathrm{Cat}_{\lfloor d/2 \rfloor}(F)$
  2. Solve *polynomial* system $\{\mathcal{F}=0\}$ to get $\mathcal{Z}eros(\mathcal{F}) \stackrel{?}{=} \{[\ell_1],\ldots,[\ell_r]\}$,
  3. Solve *linear* equations to get λ_i in $F=\sum_{i=1}^r \lambda_i \ell_i^d$
- (At least) three common approaches:
  - Gröbner bases computation (symbolic)
  - Homotopy continuation (numerical)
  - Eigenvalue/normal form methods (numerical/mixed)

$\rightsquigarrow$ Focus on the eigenvalue method approach here

Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $\{\mathcal{F} = 0\}$, compute finite set $Z = \{z_1, \ldots, z_r\} = \mathcal{Z}eros(\mathcal{F}) \subseteq \mathbb{P}^n$

- Consider ideal $J \coloneqq \langle \mathcal{F} \rangle_S = \bigoplus_{t \geq 0} J_t$, this is a graded subspace of S with

$$J_t = S_{t - \deg f_1} f_1 + \cdots + S_{t - \deg f_s} f_s \subseteq S_t$$

- For t large enough the Hilbert function $h_{S/J}(t) \coloneqq \dim_{\mathbb{C}} (S/J)_t$ is constant r
- Multiplication map for $g \in S_e$:

$$M_g \colon (S/J)_d \xrightarrow{\cdot g} (S/J)_{d+e}$$

- Under "suitable conditions" $M_h^{-1} M_g \colon (S/J)_d \to (S/J)_d$ has left eigenpairs

$$\{\, (\mathrm{ev}_{z_i}, \tfrac{g}{h}(z_i)) \mid i = 1, \ldots, r \,\}, \qquad \mathrm{ev}_{z_i}(f) = f(z_i)/h(z_i)$$

$\rightsquigarrow$ Translate problem into large eigenvalue problem, solve numerically

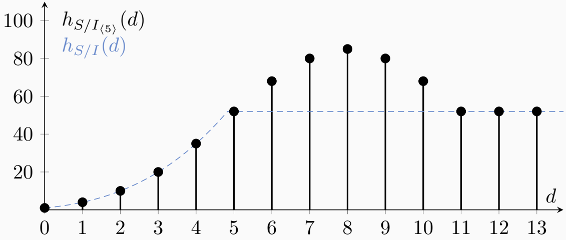
- For this need $h_{S/J}(d+e) = h_{S/J}(d) = r$, want $d, d+e$ as small as possible

Non-saturated systems are harder to solve

- Z general set of points, $I = \{\, f \in S \mid f(Z) = 0 \,\}$, then $h_{S/I}(t) = \min\{h_S(t), r\}$
  - $\rightsquigarrow$ $d = \min\{\, t \mid h_S(t) \geq r \,\}$ and $e = 1$ work.
- In general, larger saturation gap can be encountered
- Saturation gap governs *algorithmic complexity* of solving J with eigenvalue methods

Bigger example

For a general set $Z \subseteq \mathbb{P}^3$ of 52 points and $J = I_{\langle 5 \rangle}(Z) := \langle \{\, f \in S_5 \mid f(Z) = 0 \,\} \rangle_S$, we have the Hilbert function pictured below. Smallest choice: $d = 5$, $d + e = 11$.



Recap

We are lead to the following setup:

- Given a general form $F = \sum_{i=1}^{r} \ell_i^d \in \mathbb{C}[X_1, \ldots, X_n]_d$ of "small" rank r
- Decomposition is unique, want to find $Z = \{[\ell_1], \ldots, [\ell_r]\} \in \mathbb{P}^n$
- Want to solve Catalecticant polynomial system $\mathcal{F}$ using the eigenvalue method
- Is $\mathcal{Z}eros(\mathcal{F}) = Z$? With(out) multiplicities?
- What is the Hilbert function of the ideal $\langle \mathcal{F} \rangle_S \subseteq S$? When $= r$?

Running example

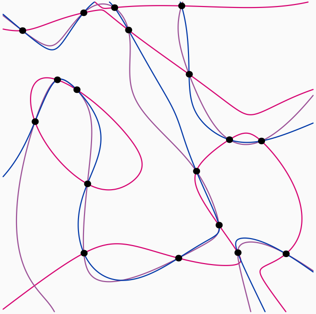
$n = 2$, $d = 10$, $r = 18$, equations $\mathcal{F}$ have degree $d/2 = 5$.

$$F = \sum_{i=1}^{18} \ell_i^{10} \in \mathbb{C}[X_0, X_1, X_2]_{10}, \qquad [\ell_i] \in \mathbb{P}(\mathbb{C}[X_0, X_1, X_2]_1) = \mathbb{P}^2$$

Example: $Z = 18$ points in the plane

t	$\dots$	3	4	5	6	7
$h_S(t)$	$\dots$	10	15	21	28	36
$h_I(t)$	$\dots$	0	0	3	10	18
$h_{I_{\langle 5\rangle}}(t)$	$\dots$	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$h_S(t)$	1	3	6	10	15	21	28	36
$h_{S/I}(t)$	1	3	6	10	15	18	18	18
$h_{S/I_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	19	18



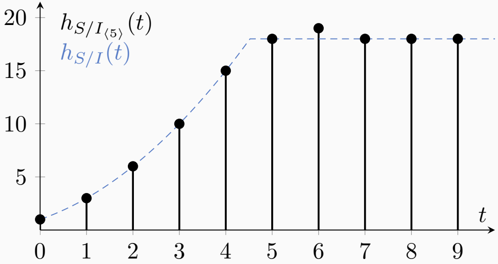


Figure 1: Three quintics $\langle q_1, q_2, q_3\rangle_{\mathbb{C}} = I_5$ passing through 18 general points.

For which forms is our algorithmic approach even possible?

- For a set of points Z consider the vanishing ideal and chopped ideal

$$I = \{\, f \in S \mid f(Z) = 0 \,\}\,, \qquad I_{\langle d \rangle} = \langle \{\, f \in S_d \mid f(Z) = 0 \,\} \rangle_S$$

- Generally $I_{\langle d \rangle} \subsetneq I$, we need $\mathcal{Z}eros(I) \overset{?}{=} \mathcal{Z}eros(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$
  multiplicities

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$. Then

$$\mathcal{Z}eros(I) = \mathcal{Z}eros(I_{\langle d \rangle}) \iff r < \binom{n+d}{n} - n \text{ or } r = 1 \text{ or } (n, r, d) = (2, 4, 2).$$

The conjectural Hilbert function

Expected syzygy conjecture (ESC)

For a general set of $r < \binom{n+d}{n} - n$ points in $\mathbb{P}^n$ the ideal $I_{\langle d \rangle}$ has Hilbert function

$$h_{S/I_{\langle d \rangle}}(t) = \begin{cases} \sum_{k \geq 0} (-1)^k \cdot \binom{n+t-kd}{n} \cdot \binom{\binom{n+d}{n} - r}{k} & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the first integer $> d$ such that the sum is $\leq r$.

- One can extract the saturation gap length from this formula
- This is always a (lexicographic) lower bound due to Fröberg
- If $W \subseteq S_d$ is a random vector subspace of dim. $\binom{n+d}{n} - r$, then this sum is the expected Hilbert function of $S/\langle W \rangle_S$ (until sum ≤ 0)

Main results

Theorem (Gesmundo, Kayser & Telen)

Conjecture (ESC) is true in the following cases:

- $r_{\max} := \binom{n+d}{n} - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\left((n+1)\binom{n+d}{n} - \binom{n+d+1}{n}\right)$ *and [*$n \leq 4$ *or* $d \gg 0$*]*
- *In a large number of individual cases in low dimension (table below).*

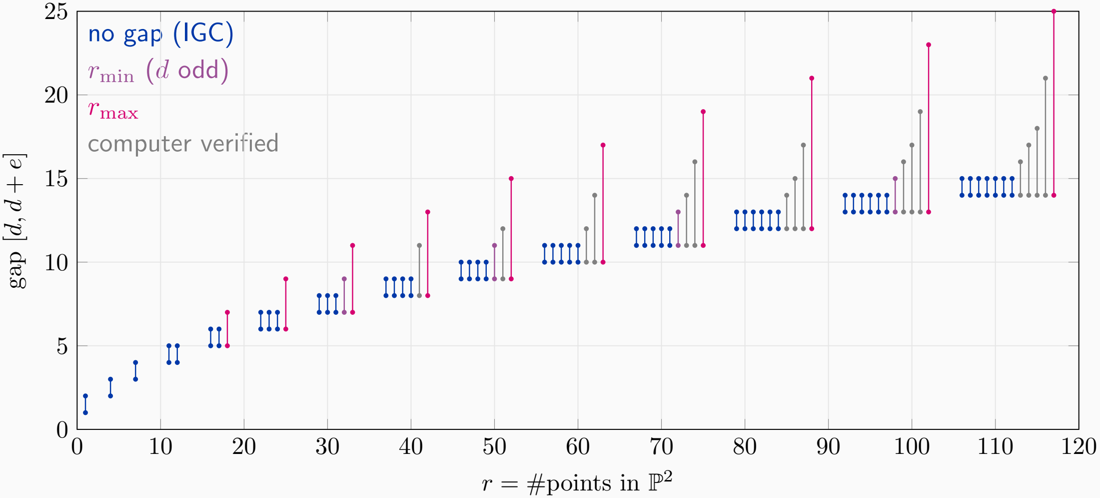
The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d\rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

n	2	3	4	5	6	7	8	9	10
r	≤ 1825	≤ 1534	≤ 991	≤ 600	≤ 447	≤ 316	≤ 333	≤ 204	≤ 259
d	≤ 58	≤ 18	≤ 9	≤ 6	≤ 4	≤ 3	≤ 3	≤ 2	≤ 2

Visualization of the saturation gaps in $\mathbb{P}^2$

- ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is



Hungry for more?

⇝ MS Tensor Decompositions and Algorithms (Thursday & Friday)

Part of MS125 Tensor Decompositions and Algorithms - Part III of III
A normal form algorithm for tensor rank decomposition

Abstract. We propose a new numerical algorithm for computing the tensor rank decomposition or canonical polyadic decomposition of higher-order tensors subject to a rank and genericity constraint. Reformulating this computational problem as a system of polynomial equations allows us to leverage recent numerical linear algebra tools from computational algebraic geometry, relying only on basic linear algebra computations and Newton refinement. Numerical experiments show that our algorithm outperforms state-of-the-art numerical algorithms by an order of magnitude in terms of accuracy, computation time, and memory consumption.

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Part of MS125 Tensor Decompositions and Algorithms - Part III of III
Algorithms and Uniqueness of Tensor Decompositions

Abstract. In contrast to matrices, tensor rank decompositions are often unique (up to trivialities). Uniqueness is useful in applications, as it corresponds to a unique interpretation of the information stored in a tensor. I will talk about recent work on (i) developing algorithms for tensor rank decompositions, and (ii) certifying that a given tensor rank decomposition is unique.

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Thank you! Questions?

arXiv:2307.02560

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[Submitted on 5 Jul 2023]

Hilbert Functions of Chopped Ideals

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A chopped ideal is obtained from a homogeneous ideal by considering only the generators of a fixed degree. We investigate cases in which the chopped ideal defines the same finite set of points as the original one-dimensional ideal. The complexity of

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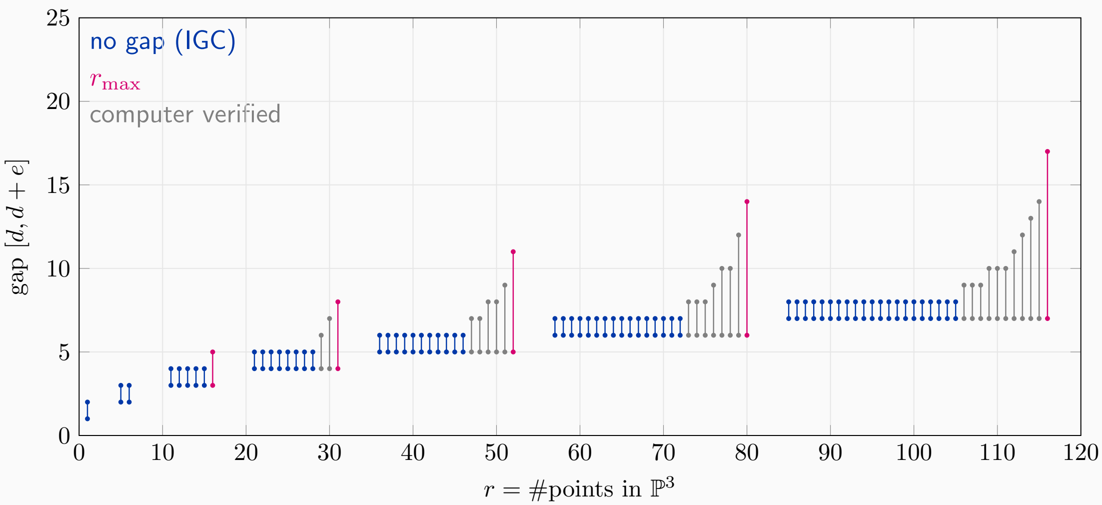
Visualization of the saturation gaps in $\mathbb{P}^3$



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