The Waring problem for polynomials

Geometry and applications

Leo Kayser

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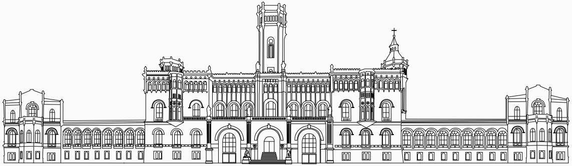
Institut für Algebraische Geometrie



Content

Waring rank and secant varieties

The Waring rank of a homogeneous form

Definition 1: (Waring rank, Waring decomposition)

Let $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ be a form. The *Waring rank* $\mathrm{WR}(F)$ is the least $r \in \mathbb{N}_0$ such that there exists a decomposition

$$F = \lambda_1 L_1^d + \cdots + \lambda_r L_r^d, \qquad L_1, \ldots, L_r \in \mathbb{C}[\underline{x}]_1 \text{ linear forms, } \lambda_i \in \mathbb{C}.$$

Any such expression is called a *Waring decomposition* of F.

This notion is

- independent of the number of variables of the ambient space
- invariant under scaling with $\lambda \in \mathbb{C}^\times$, i. e. $\mathrm{WR}(\lambda F) = \mathrm{WR}(F)$
- invariant under changes of coordinates, i. e. $\mathrm{WR}(F \circ A) = \mathrm{WR}(F)$, $A \in \mathrm{GL}_{n+1}(\mathbb{C})$

Examples and leading questions

Example 2

- $\mathrm{WR}(x_1^d + \cdots + x_k^d) = k$
- If $F(x) = x^\mathsf{T} A x$ for $A \in \mathrm{Sym}_{n+1}(\mathbb{C})$, then $\mathrm{WR}(F) = \operatorname{rank} A$
- Let $d \geq 3$. $\mathrm{WR}(x_0 x_1^{d-1}) = d$, although

$$x_0 x_1^{d-1} = \frac{1}{d} \cdot \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left((\varepsilon x_0 + x_1)^d - x_1^d\right)$$

- Is $\mathrm{WR}(F)$ always finite? Does the set of forms of rank r have a *nice* structure?
- What is the Waring rank of monomials or other basic families of forms?
- What can be said about the maximal rank? Or the rank of a general form?
- Are there (efficient) algorithms for the Waring rank?

Powers of linear forms as a projective variety

Fix $n, d \in \mathbb{N}_+$ and $N := \binom{n+d}{d} - 1$. Consider the morphism

$$\nu_d \colon \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_1) \to \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_d) =: \mathbb{P}^N, \qquad [L] \mapsto [L^d],$$

this is (up to a change of coordinates) the closed embedding associated to $\mathcal{O}_{\mathbb{P}^n}(d)$.

Definition 3: (Veronese embedding, Veronese variety)

The map ν_d is called the *Veronese embedding*, its image is the *Veronese variety* $V^{d,n} \subseteq \mathbb{P}^N$.

Observation: $V^{d,n}$ is a closed subvariety of $\mathbb{P}^N$ not contained in hyperplane.

Higher secant varieties parameterize the Waring rank

Definition 4: (Higher secant variety)

Let $X \subseteq \mathbb{P}^N$ be a projective variety. Consider the following subset of $\mathbb{P}^N$:

$$\sigma_s^\circ X := \bigcup_{p_1,\ldots,p_s \in X} \langle p_1, \ldots, p_s \rangle_{\mathbb{P}}\,, \qquad \sigma_s X := \overline{\sigma_s^\circ X}.$$

$\sigma_s X$ is called the *s-th higher secant variety* of X.

Consequence: We have

$$\{\, [F] \in \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) \mid \mathrm{WR}(F) \leq s \,\} = \sigma_s^\circ V^{d,n}.$$

In particular $\mathrm{WR}(F) \leq \binom{n+d}{d}$ for any form F.

Small detour: Constructible sets

Definition 5: (Constructible set)

A subset of a variety X is *constructible* if it is a finite union of locally closed sets

$$A = \bigcup_{i=1}^{m} C_i \cap O_i, \qquad C_i \text{ closed}, \ O_i \text{ open}.$$

Important properties:

- If $A, B \subseteq X$ are constructible, then so are $A \cup B$, $A \cap B$, $A \setminus B$
- (Chevalley) If $X \to Y$ is a morphism of varieties and $A \subseteq X$ is constructible, then $f(A) \subseteq Y$ is also constructible
- If $A \subseteq \mathbb{C}^n$ is constructible, then $\overline{A}^{\mathbb{C}} = \overline{A}$ (Euclidean vs. Zariski topology)

Waring rank is a constructible property

Lemma 6 If $X \subseteq \mathbb{P}^N$ is a variety, then $\sigma_s^\circ X$ is a constructible irreducible set.

Consequence: The following sets are irreducible and constructible:

$$W_{\leq s} = \{ F \in \mathbb{C}[x_0, \ldots, x_n]_d \mid \mathrm{WR}(F) \leq s \},$$
$$W_s = \{ F \in \mathbb{C}[x_0, \ldots, x_n]_d \mid \mathrm{WR}(F) = s \}.$$

Definition 7: (Border rank)

The *border Waring rank* of $F \in \mathbb{C}[\underline{x}]_d$ is $\underline{\mathrm{WR}}(F) = \min \left\{ r \in \mathbb{N}_0 \;\middle|\; F \in \overline{W_{\leq r}} \right\}$.

The closure $\overline{W_{\leq s}}$ consists of limits of forms of rank $\leq r$, e. g. $x_0 x_1^{d-1} \in \overline{W_{\leq 2}}$.

Expectation vs. reality

Lemma 8: The expected dimension

Let $X \subseteq \mathbb{P}^N$ be a projective variety not contained in a hyperplane. Then

$$\dim \sigma_s X \leq \min\{s \cdot \dim X + s - 1, N\} =: \operatorname{expdim} \sigma_s X.$$

Definition 9: (s-defect of secant varieties)

The difference $\delta_s := \operatorname{expdim} \sigma_s X - \dim \sigma_s X$ is the *s-defect* of X.
If $\delta_s > 0$ then X is said to be *s-defective*.

- Curves are never s-defective
- The Veronese surface $V^{2,2} \subseteq \mathbb{P}^5$ is 2-defective

How to calculate $\dim \sigma_s X$

Theorem 10: (Terracini's first lemma)

For a general collection of points $p_1, \ldots, p_s \in X$ and a general point $q \in \langle p_1, \ldots, p_s \rangle_{\mathbb{P}}$ we have

$$T_q \sigma_s(X) = \langle T_{p_1} X, \ldots, T_{p_s} X \rangle_{\mathbb{P}} .$$

Lemma 11: The tangent space of $V^{d,n}$

The tangent space $T_{[L^d]} V^{d,n}$ is the subspace

$$T_{[L^d]} V^{d,n} = \left\{ [L^{d-1} F] \;\middle|\; F \in \mathbb{C}[\underline{x}]_1 \right\} \subseteq \mathbb{P}(\mathbb{C}[\underline{x}]_d).$$

$\sigma_s V^{d,n}$ has (mostly) the expected dimension

Theorem 12: (Alexander-Hirschowitz [BO08])

Let $n, d, s \geq 1$, then we have

$$\dim \sigma_s V^{d,n} = \operatorname{expdim} \sigma_s V^{d,n} = \min \left\{ sn + s - 1, \binom{n+d}{d} - 1 \right\}$$

with the following list of exceptions:

d	n	s	δ_s	$\dim \sigma_s V^{d,n}$
2	≥ 2	$2 \dots n$	$\binom{s}{2}$	$sn + s - 1 - \binom{s}{2}$
3	4	7	1	33
4	2	5	1	14
4	3	9	1	33
4	4	14	1	68

The generic Waring rank

The *big Waring problem* asks for the rank $G(n,d)$ of a general form, i. e. the rank of a dense open set of forms $F \in U \subseteq \mathbb{C}[x_0, \dots, x_n]_d$.

Corollary 13: (The solution to the big Waring problem)

$G(n,d) = \lceil \frac{1}{n+1} \binom{n+d}{d} \rceil$ with the following list of exceptions

d	n	$G(n,d)$
2	$\forall$	$n+1$
3	4	8
4	2	6
4	3	10
4	4	15

The maximum Waring rank

The *little Waring problem* asks for the largest possible rank $g(n,d)$ of a form $F \in \mathbb{C}[x_0,\dots,x_n]_d$.

- $g(1,d) = d$ (attained by $x_0 x_1^{d-1}$)
- $g(n,2) = n+1$ (attained by $x_0^2 + \cdots + x_n^2$)
- Upper bound by Ballico & De Paris [BD17]:

$$g(n,d) \leq \binom{n+d-1}{n} - \binom{n+d-5}{n-2} - \binom{n+d-6}{n-2}$$

- (Asymptotically) better bound by Blekherman & Teitler [BT14]:

$$G(n,d) \leq g(n,d) \leq 2 \cdot G(n,d)$$

Apolarity and the rank of monomials

The apolarity pairing

Let $T := \mathbb{C}[x_0, \dots, x_n]$, $X_i := \frac{\partial}{\partial x_i}$, $S := \mathbb{C}[X_0, \dots, X_n]$ and consider the pairing

$$S_i \times T_j \to T_{j-i}, \qquad X^{\boldsymbol{\alpha}} \circ x^{\boldsymbol{\beta}} := \begin{cases} \frac{\boldsymbol{\beta}!}{(\boldsymbol{\beta}-\boldsymbol{\alpha})!} x^{\boldsymbol{\beta}-\boldsymbol{\alpha}} & \text{if } \boldsymbol{\alpha} \leq \boldsymbol{\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 14: Properties of the apolarity paring

- T is a S-module with $\circ$ as scalar multiplication.
- $S_d \times T_d \to \mathbb{C}$ is a perfect pairing for $d \geq 0$.
- If $L = a_0x_0 + \dots + a_nx_n$ is a linear form and $f \in S_d$, then

$$f \circ L^d = d! \cdot f(a_0, \dots, a_n).$$

Hence we can view S as a ring of functions on $\mathbb{P}(T_1) \cong \operatorname{Proj} S$.

From differential operators to forms: Inverse systems

Definition 15: (Inverse system)

For a homogeneous ideal $I \subseteq S$, the *inverse system* is

$$I^{-1} := \{ F \in T \mid \partial \circ F = 0 \ \forall \partial \in I \} .$$

Lemma 16 Let $I, J \subseteq S$ be homogeneous ideals, then

- $(I^{-1})_d = (I_d)^\perp := \{ F \in T_d \mid \partial \circ F = 0 \ \forall \partial \in I_d \}$
- $I \subseteq J \implies J^{-1} \subseteq I^{-1}$
- $(I + J)^{-1} = I^{-1} \cap J^{-1}$, $(I \cap J)^{-1} = I^{-1} + J^{-1}$
- $\dim_{\mathbb{C}} I_d^{-1} = \dim_{\mathbb{C}}(S/I)_d = \dim_{\mathbb{C}} S_d - \dim_{\mathbb{C}} I_d$

From forms to differential operators: Apolar ideals

Definition 17: (Apolar ideal)

For a form $F \in T_d$, its *apolar ideal* is the homogeneous ideal

$$F^\perp := \{\, \partial \in S \mid \partial \circ F = 0 \,\}.$$

Example 18 Consider $F = L^d \in T_d$.

- The apolar ideal $I := F^\perp$ is the vanishing ideal of $[L] \in \mathbb{P}(T_1)$.
- Conversely, one has $I_d^{-1} = \mathbb{C} \cdot L^d$.

A characterization of the Waring rank

Theorem 19: (Apolarity Lemma)

Let $L_1, \dots, L_s \in T_1$ be linear forms and $\mathbb{X} = \{[L_1], \dots, [L_s]\} \subseteq \mathbb{P}(T_1)$. Then for a form $F \in T_d$ the following are equivalent:

(i) $F = \lambda_1 L_1^d + \cdots + \lambda_s L_s^d$ for some $\lambda_i \in \mathbb{C}$;

(ii) $I(\mathbb{X}) \subseteq F^\perp$.

Corollary 20

Let $0 \neq F \in T$ be a form, then

$$\mathrm{WR}(F) = \min\left\{ r \in \mathbb{N}_+ \;\middle|\; F^\perp \text{ contains the ideal of a set of } r \text{ distinct points} \right\}.$$

The Waring rank of monomials

Theorem 21: (Carlini, Catalisano & Geramita [CCG12])

Let $x_0^{d_0} \cdots x_n^{d_n} \in \mathbb{C}[\underline{x}]$ be a monomial. After renaming the variables we may assume $1 \leq d_0 \leq \cdots \leq d_n$. Then

$$\mathrm{WR}(x_0^{d_0} \cdots x_n^{d_n}) = \frac{1}{d_0 + 1} \prod_{i=0}^{n} (d_i + 1).$$

Example 22 A Waring decomposition of $F = x_0 \cdots x_n$ is given by

$$x_0 \cdots x_n = \frac{1}{2^n n!} \sum_{\xi \in \{\pm 1\}^n} \xi_1 \cdots \xi_n \cdot (x_0 + \xi_1 x_1 + \cdots + \xi_n x_n)^n.$$

The symmetric Strassen conjecture

Conjecture 23 If $F_j \in \mathbb{C}[x_{0,j}, \ldots, x_{n_j,j}]_d$, $j = 1, \ldots, m$, $d \geq 2$ are forms in disjoint sets of variables, then their sum in $\mathbb{C}[\{ x_{i,j} \mid i, j \}]_d$ has Waring rank

$$\mathrm{WR}(F_1 + \cdots + F_m) = \mathrm{WR}(F_1) + \cdots + \mathrm{WR}(F_m).$$

Carlini et al. [Car+15] showed this to be true if each F_i is of one of the following:

- F_i is a monomial;
- F_i is a form in ≤ 2 variables;
- $F_i = x_0^a(x_1^b + \cdots + x_n^b)$ or $F_i = x_0^a(x_0^b + x_1^b + \cdots + x_n^b)$ with $a + 1 \geq b$;
- $F_i = x_0^a(x_1^b + x_2^b)$ or $F_i = x_0^a(x_0^b + x_1^b + x_2^b)$;
- $F_i = x_0^a G(x_1, \ldots, x_n)$, where $G^\perp = (g_1, \ldots, g_n)$ is a complete intersection ideal and $\deg(g_j) > a$ for $j = 1, \ldots, n$;
- $F_i = \det([x_j^k]_{j,k=0}^n)$ is a Vandermonde determinant.

Elementary symmetric polynomials

Recall the *elementary symmetric polynomials* $e_{n,d} = \sum_{1 \le i_1 < \cdots < i_d \le d} x_{i_1} \cdots x_{i_d}$.

Theorem 24: (Lee [Lee16])

For $d = 2k+1$ odd, $n \ge d$, we have a Waring decomposition

$$2^{d-1} d! e_{n,d} = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| \le k}} (-1)^{|I|} \binom{n-k-|I|-1}{k-|I|} \cdot \left(\delta(I,1)x_1 + \cdots + \delta(I,n)x_n\right)^d,$$

where $\delta(I,i) = -1$ if $i \in I$, $+1$ otherwise. In particular $\mathrm{WR}(e_{n,d}) = \sum_{i=0}^{\frac{d-1}{2}} \binom{n}{i}$.
A similar decomposition is possible for d even, but this is known to be suboptimal.

Applications to computer science

Counting simple closed walks

Let $G = (V, E)$ be a directed graph, $V = \{v_1, \ldots, v_n\}$.

Definition 25

(i) A *walk* of length d in G is a sequence $w = (v_{i_0}, \ldots, v_{i_d})$ with $(v_{i_{j-1}}, v_{i_j}) \in E$ for $j = 1, \ldots, d$.

(ii) If $i_d = i_0$, then w is a *closed walk*. If, additionally, all nodes in w are pairwise distinct (apart from $v_{i_0} = v_{i_d}$), then w is called a *simple cycle*.

Problem 26 Describe an algorithm which on input $\langle G, d \rangle$ calculates the number of simple cycles in G of length d.

The graph walk polynomial

Consider the *symbolic adjacency matrix* and the *graph walk polynomial*

$$A_G := [a_{ij}] \in \mathrm{Mat}_n(\mathbb{C}[\underline{x}]_1), \quad a_{ij} = \begin{cases} x_i & \text{if } (v_i, v_j) \in E; \\ 0 & \text{otherwise,} \end{cases} \qquad F_G := \mathrm{tr}(A_G^d) \in \mathbb{C}[\underline{x}]_d.$$

Lemma 27: Extracting the number of simple cycles from F_G

(i) The terms of F_G represent closed walks of length d in G:

$$F_G = \sum_{\text{closed walks } (v_{i_0}, \ldots, v_{i_d})} x_{i_0} \cdots x_{i_{d-1}}.$$

(ii) The number of simple cycles of length d in G is given by

$$e_{n,d}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) F_G.$$

Apolarity - Reprise

Lemma 28 Let $F \in \mathbb{C}[x_1, \ldots, x_n]_d$, $g \in \mathbb{C}[X_1, \ldots, X_n]_d$.

(i) We can "switch" the roles of F and g in the apolarity action, i. e. we have the identity

$$g(\underline{X}) \circ F(\underline{x}) = F(\underline{X}) \circ g(\underline{x}).$$

(ii) If $F = \lambda_1 L_1^d + \cdots + \lambda_s L_s^d$, where $L_i = c_{i,1}x_1 + \cdots + c_{i,n}x_n \in \mathbb{C}[\underline{x}]_1$, then

$$g \circ F = d! \cdot \sum_{i=1}^{r} \lambda_i g(c_{i,1}, \ldots, c_{i,n}).$$

A simple formula

Consequence: If $e_{n,d} = \sum_{i=1}^{s} \lambda_i L_i^d$, $L_i = c_{i,1}X_1 + \cdots + c_{i,n}X_n$, then for *any* G we get

$$\#\{\text{simple cycles of length } d \text{ in } G\} = d! \sum_{i=1}^{s} \lambda_i F_G(c_{i,1}, \ldots, c_{i,n}).$$

Applying Lee's power sum decomposition of $e_{n,d}$ (in the case d odd) yields the formula

$$\#\{\text{simple length } d \text{ cycles in } G\} =$$

$$\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| \leq \lfloor d/2 \rfloor}} \frac{(-1)^{|I|}}{2^{d-1}} \binom{n - \lfloor d/2 \rfloor - |I| - 1}{\lfloor d/2 \rfloor - |I|} \cdot F_G\big(\delta(I,1), \ldots, \delta(I,n)\big).$$

The best solution?

Corollary 29

This formula yields a $\binom{n}{\lfloor d/2 \rfloor} \cdot \mathrm{poly}(n)$ time and $\mathrm{poly}(n)$ space algorithm for counting simple cycles.

In some sense this is optimal:

Theorem 30: (Pratt [Pra18, Thm. 6])

Fix $g \in \mathbb{C}[\underline{x}]$ and let $F \in \mathbb{C}[\underline{x}]$ be given as a black-box.
The minimum number of queries to F needed to compute $g(\frac{\partial}{\partial x})F$ is $\mathrm{WR}(g)$.

Definition 31: (Catalecticant matrix)

Let $F = \sum_{i=0}^{d} a_i \binom{d}{i} x_0^i x_1^{d-i}$ be a binary form. Its *Catalecticant matrix* is

$$\mathrm{Cat}_{r,d-r}(F) := \begin{bmatrix} a_0 & a_1 & \dots & a_r \\ a_1 & a_2 & \dots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r} & \dots & a_d \end{bmatrix}.$$

It is the matrix representing the linear map

$$S_r \to T_{d-r}, \qquad g \mapsto g \circ F.$$

An algorithm for the Waring rank of binary forms

Sylvester's algorithm

Require: A binary form $0 \neq F = \sum_{i=0}^{d} a_i \binom{d}{i} x_0^i x_1^{d-i} \in \mathbb{C}[x_0, x_1]_d$.

Ensure: $r = \mathrm{WR}(F)$, $F = \sum_{j=1}^{r} \lambda_i L_i^d$ a Waring decomposition.

1: $r \leftarrow 1$.
2: **while** $\mathrm{rank}\,\mathrm{Cat}_{r,d-r}(F)$ is maximal **do**
3:     $r \leftarrow r + 1$
4: **end while**
5: Take any nontrivial element $0 \neq F_0 \in \ker \mathrm{Cat}_{r,d-r}(F)$.
6: Compute the roots $(\alpha_i, \beta_i) \in \mathbb{C}^2$ of F_0, $i = 1, \ldots, r$.
7: **if** the roots are not distinct in $\mathbb{P}(\mathbb{C}^2)$ **then**
8:     go to step 2   //i. e. increase r further
9: **else**
10:     Construct the set of linear forms $\{L_i = \alpha_i x_0 + \beta_i x_1\}$.
11:     Solve the linear system of equations $F = \sum_{i=1}^{r} \lambda_i L_i^d$.
12:     **return** the Waring decomposition $F = \sum_{i=1}^{r} \lambda_i L_i^d$.
13: **end if**

The Waring problem is NP-hard

Consider the formal languages

$$\texttt{WARING\_RANK}_{\mathbb{C}/\mathbb{Q}} = \{ \langle F, r \rangle \mid r \in \mathbb{N}_0,\ F \in \mathbb{Q}[x_1, \ldots, x_n]_d,\ \mathrm{WR}(F) \leq r \},$$

$$\texttt{NULLSTELLENSATZ}_{\mathbb{C}/\mathbb{Q}} = \left\{ \langle f_1, \ldots, f_m \rangle \;\middle|\; \begin{array}{c} f_1, \ldots, f_m \in \mathbb{Q}[T_1, \ldots, T_n] \\ \text{have a common root in } \mathbb{C}^n \end{array} \right\}.$$

Theorem 32: (Shitov [Shi16])

The languages $\texttt{WARING\_RANK}_{\mathbb{C}/\mathbb{Q}}$ and $\texttt{NULLSTELLENSATZ}_{\mathbb{C}/\mathbb{Q}}$ are polynomial-time equivalent under many-one reductions.
In particular, the problem $\texttt{WARING\_RANK}_{\mathbb{C}/\mathbb{Q}}$ is NP-hard.

Thank you! Any questions?

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