What is a Hilbert function?

Seminar day on Algebra, Geometry and Computation at CWI

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March 9, 2023

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The impossible quiz

Figure 1: Which of the following configurations of three points is more ✨special✨?

The impossible quiz

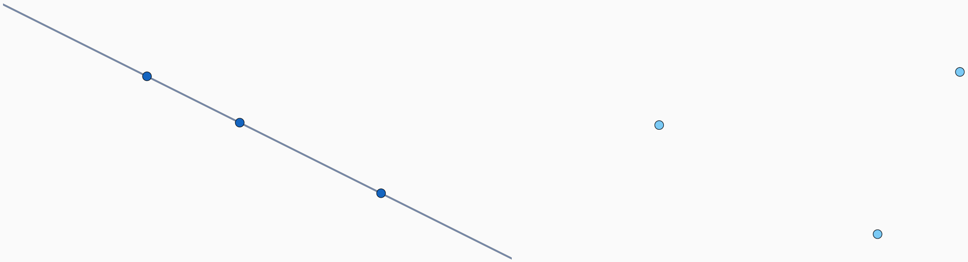

Figure 1: Which of the following configurations of three points is more ✨special✨?

The impossible quiz 2

Figure 2: Which of the following configurations of six points is more ✨special✨?

The impossible quiz 2

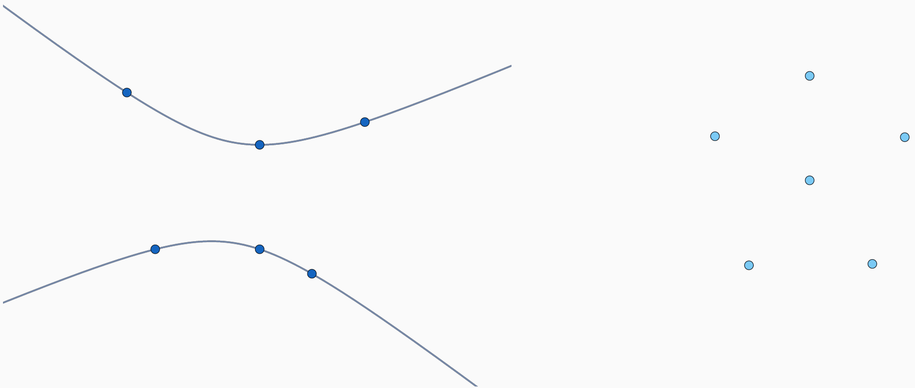

Figure 2: Which of the following configurations of six points is more ✨special✨?

Polynomials vanishing on points

- Consider a finite set of points $X \subseteq \mathbb{C}^n$

⇝ **Question:** How many polynomials of degree $\leq m$ vanish on X?

- Here “many” means the dimension of the vector space

$$I_{\leq m}(X) = \{\, f \mid \deg(f) \leq m \text{ and } f(x) = 0 \text{ for } x \in X \,\}$$

- Example of three points in the plane

m	1	2	3	4	5	6	7	8	9	10
$\dim I_{\leq m}(\cdots)$	1	3	7	12	18	25	33	42	52	63
$\dim I_{\leq m}(\because)$	0	3	7	12	18	25	33	42	52	63

Graded vector spaces and their Hilbert functions

- A *graded vector space* (over $\mathbb{C}$) is a vector space V with a decomposition into finite vector spaces

$$V = \bigoplus_{d \geq 0} V_d = V_0 \oplus V_1 \oplus V_2 \oplus \dots$$

Definition (Hilbert function)

The *Hilbert function* of a graded vector space is $h_V : \mathbb{N} \to \mathbb{N}$, $h_V(m) := \dim_{\mathbb{C}} V_m$.

- **Important example:** The *polynomial ring* $S = \mathbb{C}[X_1, \dots, X_n]$,

$$S_d = \left\{ f = \sum_{|\boldsymbol{\alpha}|=d} f_{\boldsymbol{\alpha}} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \;\middle|\; f_{\boldsymbol{\alpha}} \in \mathbb{C} \right\}, \qquad |\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$$

- $h_S(d) = \#\{\text{ monomials of degree } d\ \} = \binom{d+n-1}{n} = \frac{(d+n-1)(d+n-2)\cdots d}{n!}$

From the affine to the projective world

- Compactify $\mathbb{C}^n$ into *projective space*

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim, \qquad x \sim y \text{ iff } y = \lambda x, \quad \lambda \in \mathbb{C}^\times$$

  with the inclusion $\mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (1 : x_1 : \cdots : x_n) \in \mathbb{P}^n$
- $\mathbb{P}^n$ is "nicer" than $\mathbb{C}^n$, e.g. any system of n polynomials has solutions in $\mathbb{P}^n$
- For each $d \geq 0$ we have a bijection $\mathbb{C}[X_1, \dots, X_n]_{\leq d} \longleftrightarrow \mathbb{C}[X_0, \dots, X_n]_d$

$$f = \sum_{|\boldsymbol{\alpha}| \leq d} f_{\boldsymbol{\alpha}} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mapsto f^{\mathrm{h}} = \sum_{|\boldsymbol{\alpha}| \leq d} f_{\boldsymbol{\alpha}} X_0^{d-|\boldsymbol{\alpha}|} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

- f vanishes on $(x_1, \dots, x_n) \in \mathbb{C}^n \iff f^{\mathrm{h}}$ vanishes on $(1 : x_1 : \cdots : x_n) \in \mathbb{P}^n$

⇝ For $X \subseteq \mathbb{P}^n$ investigate the spaces $I(X)_d = \{ f \in S_d \mid f(x) = 0 \; \forall x \in X \}$!

The Hilbert function of a projective set

Definition

Let $X \subseteq \mathbb{P}^n$ be a set, $S = \mathbb{C}[X_0, \ldots, X_n]$.

The *homogeneous vanishing ideal* of X is the graded vector subspace

$$I(X) = \bigoplus_{d \geq 0} I(X)_d \subseteq S, \qquad I(X)_d := \{ f \in S_d \mid f(x) = 0 \text{ for all } x \in X \}.$$

The *homogeneous coordinate ring* of X is the graded quotient $S_X := S/I(X)$.

The *Hilbert function* of X is $h_X(m) := h_{S_X}(m) = h_S(m) - h_{I(X)}(m)$.

Example: Let $X, X' \subseteq \mathbb{P}^2$ be six points as in quiz 2; X lying on a conic:

m	0	1	2	3	4	5	6	7
$h_{I(X)}(m)$	0	0	1	4	9	15	22	30
$h_{I(X')}(m)$	0	0	0	4	9	15	22	30

m	0	1	2	3	4	5	6	7
$h_X(m)$	1	3	5	6	6	6	6	6
$h_{X'}(m)$	1	3	6	6	6	6	6	6

The Hilbert function of points becomes constant

The Hilbert functions of X and X' are distinct, but eventually agree with the number of points... but why? 🤔

- Elements of $(S_X)_d$ are restrictions of homogeneous polynomials $f \in S_d$ to X
- If $\#X = r$, then $\dim \mathrm{Maps}(X, \mathbb{C}) = r$

$\rightsquigarrow$ $h_X(m) \leq r$ for all $m \geq 0$

- If $d \gg 0$, then all functions can be realised

$\rightsquigarrow$ $h_X(m) = r$ for $m \gg 0$

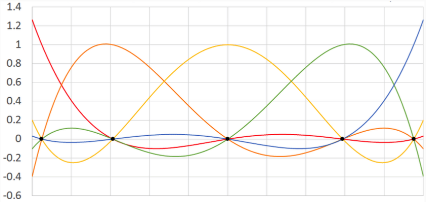

Figure 3: Lagrange polynomials

Conclusion: h_X knows the number of points and some geometry of X!

Let's step up the dimension!

- A *plane curve* is the vanishing locus of a polynomial $f \in \mathbb{C}[X_0, X_1, X_2]_d$,

$$C = \mathcal{V}(f) = \{\, x \in \mathbb{P}^2 \mid f(x) = 0 \,\}$$

- The degree of C is $\deg C := \deg(f) = d$
- C is *smooth* if the partial derivatives $\frac{\partial f}{\partial X_j}$ have no common zero in $\mathbb{P}^n$
- A smooth plane curve C is a *compact Riemann surface* (complex 1-dim'l)
- The number of holes in the (real) surface C is the *genus* $g(C)$

Figure 4: Compact Riemann surfaces of genus $g = 0, 1, \ldots$

The Hilbert function of plane curves

- By Hilbert's Nullstellensatz $I(C) = \{ f \cdot g \mid g \in S \}$
- In particular $h_{I(C)}(m) = h_S(m-d) = \binom{m-d+2}{2}$ and

$$h_C(m) = h_S(m) - h_{I(C)}(m) = \binom{m+2}{2} - \binom{m-d+2}{2}$$
$$= \cdots = dm + 1 - \frac{(d-1)(d-2)}{2} = dm + 1 - g(C)$$

Theorem

Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d and genus g. Then for m large enough the Hilbert function agrees with the linear function

$$h_C(m) = d \cdot m + (1-g), \qquad m \gg 0$$

Projective varieties and two important invariants

- A *projective variety* is the vanishing set of a set of homogeneous polynomials

$$X = \mathcal{V}(f_1, \ldots, f_s) \subseteq \mathbb{P}^n$$

- The *dimension* of X is its dimension as a complex manifold
- If $\dim X = k$, then X intersects any linear subspace $L \subseteq \mathbb{P}^n$ of dimension $n - k$
- A *general* linear space of dimension $n - k$ intersects X in a finite set of $d > 0$ points, d is the *degree* of X

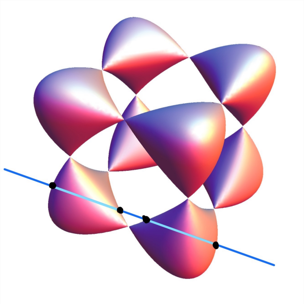

Figure 5: The Goursat surface

The big picture

Theorem (Existence of the Hilbert polynomial)

For a projective variety $X \subseteq \mathbb{P}^n$ there exists a polynomial $P_X(t) \in \mathbb{Q}[t]$ such that

$$h_X(m) = P_X(m), \qquad m \gg 0.$$

This Hilbert polynomial has the following properties:

1. $\deg(P_X) = \dim X =: k$;
2. $k! \cdot P_X$ *has integer coefficients;*
3. *the leading term of* P_X *is* $\frac{\deg X}{k!} t^k$.

- This theorem applies more generally to finitely generated graded S-modules
- The constant term $P_X(0)$ is related to the *arithmetic genus* of X

A surprising connection to combinatorics

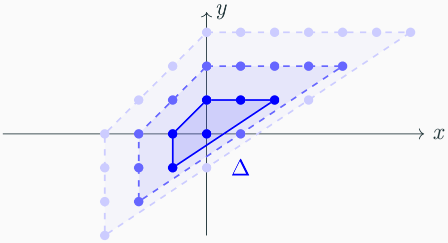

Figure 6: An integral polytope Δ and its integral points.

Theorem (The Ehrhart polynomial)

The map $L(\Delta, m) := \#(m\Delta \cap \mathbb{Z}^n)$ *is a degree* $\dim(\Delta)$ *polynomial for* $m \geq 0$*.*
Its leading coefficient is $\mathrm{vol}(\Delta)$*.*

This can be proven by relating $L(\Delta, m)$ to the Hilbert function of a graded module!

Outlook 👀

1. At which $m \in \mathbb{N}$ does $h_X(m)$ *actually* agree with $P_X(m)$?

$\rightsquigarrow$ Conditions on regularity of X

2. If $X, Y \subseteq \mathbb{P}^n$ share $P_X = P_Y$, what other properties do they share?

$\rightsquigarrow$ The *Hilbert scheme* parametrizes such varieties X with fixed P_X

3. What happens to h_X when you restrict your attention to a subspace of $I(X)$?

$\rightsquigarrow$ Current project with Simon Telen & Fulvio Gesmundo on non-saturated ideals of general collections of points

Thank you! Questions?

Image credit

- Figure 1, 2: Created using GeoGebra https://www.geogebra.org/
- Figure 3: https://www.researchgate.net/figure/Lagrange-polynomials-for-5-solution-points-N-5_fig28_314236855
- Figure 4: http://www.map.mpim-bonn.mpg.de/File:Surfaces.png
- Figure 5: http://www.grad.hr/geomteh3d/Plohe/plohe2_eng.html
- Figure 6: Based on code from https://arxiv.org/abs/2208.08179